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The high energy semiclassical asymptotics of loci of roots of fundamental solutions for polynomial potentials

Stefan Giller

Jan Dlugosz Academy in Czestochowa Institute of Physics ul. Armii Krajowej 13/15, 42-200 Czestochowa, Poland

E-mail: sgiller@uni.lodz.pl

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Abstract

In the case of polynomial potentials all solutions to 1D Schrödinger equations are entire functions totally determined by loci of their roots and their behaviour at infinity. In this paper a description of the first of the two properties is given for fundamental solutions for the high complex energy limit when the energy is quantized or not. In particular due to the fact that the limit considered is semiclassical it is shown that loci of roots of fundamental solutions are collected on selected Stokes lines (called exceptional) specific for the solution considered and are distributed along these lines in a specific way. A stable asymptotic limit of loci of zeros of fundamental solutions on their exceptional Stokes lines have island forms and there are infinitely many of such roots islands on exceptional Stokes lines escaping to infinity and a finite number of them on exceptional Stokes lines which connect pairs of turning points. The results obtained for asymptotic roots distributions of fundamental solutions in the semiclassical high (complex) energy limit are of a general nature for polynomial potentials.

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1. Introduction

As it is already well known [1, 2] the fundamental solutions (FS) [3] have appeared to play a main role in one-dimensional quantum mechanics (or in a multi-dimensional one allowing a reduction to the one dimension) with analytic potentials, i.e. polynomial, meromorphic etc. In particular they allow us to solve all basic problems typical for the field—eigenvalue problems, scattering problems, problems of decaying and JWKB and adiabatic limits [4]. They are exceptional also among all the solutions to the corresponding Schrödinger equations because of their property of being Borel summable for the polynomial or meromorphic potentials [5]. The latter property allows us to recover these solutions from their JWKB series, both exactly and approximately if the semiclassical series is abbreviated in the latter case. Finally they allow us to use in effective calculations their approximations by JWKB formulae having these approximations under full control.

Not surprising therefore that the fundamental solutions are also very important for mathematicians, used by them under the name of subdominant solutions, to study properties of solutions to Schrödinger equations (SE) for polynomial and meromorphic potentials [6]. Among many problems to be under consideration is that of loci of zeros of solutions to SE as a function of coefficients of polynomial or meromorphic (rational) potentials (see, for example [7]). This problem is also important from the physical point of view to mention the known relation between a number of real zeros of the quantized solution to SE and a number of an energy level corresponding to the solution. In a recent paper of Bender *et al* [8] a method of looking for zeros of eigenfunctions of eigenvalue problems with non-Hermitian potentials was suggested as a tool for checking a possible completeness of the full set of such eigenfunctions.

In another recent paper of Eremenko *et al* [9] the high energy limit has been considered to study this problem for a quantized energy in polynomial potentials. The authors have shown that in this limit the problem simplifies greatly so that it can be standardized and the loci of zeros of the corresponding quantized FS's (i.e. these which are a solution to an eigenvalue problem) are exactly on Stokes lines, called exceptional by the authors, of a Stokes graph (or global Stokes lines—a name used by mathematicians) corresponding to a considered problem.

Another semiclassical limit has been considered by Hezari [10] who investigated the problem of complex zeros of eigenfunctions of SE with real polynomial potentials of even degree in the limit $\hbar \to 0$, where $\hbar$ is the Planck constant, while the energy parameter E was kept fixed.

However, Hezari's eigenfunctions are as such a problem of the quantized Planck constant (being by this a little bit unphysical). In fact the two cases, i.e. the energy quantized while the Planck constant is kept fixed and the energy kept fixed but the Planck constant is quantized have two different semiclassical limits for the quantized parameters, i.e. the high energy limit and the small $\hbar$-limit lead to different behaviour of the corresponding Stokes graphs and to different sets of eigenfunctions. Nevertheless, since both these limits are of the same semiclassical nature at least mathematically it is therefore not surprising that Hezari's results of complex zeros eigenfunctions problem are similar to those of Eremenko *et al*. We have discussed and generalized Hezari's results in a separate paper [11]

In this paper we would like both to generalize the results of Eremenko *et al* [9] and to make the corresponding theorems more precise in the following aspects:

(1) to show that they are valid for unquantized FS's in the considered limit so that quantized cases can be seen as particular results of these general ones.
(2) to show how the quantization procedure modifies unquantized zeros distributions of FS's.
(3) to show that the exceptional Stokes lines of Eremenko *et al* [9] can be identified with the boundaries of the canonical domains corresponding to FS's.
(4) to find the limit distributions of zeros of FS's along exceptional SL's using the explicit form of FS's and their high energy semiclassical limit.

In fact our analysis provides a full description of the root distribution problem of FS's to SE with polynomial potentials in the high (complex) energy limit.

The paper is organized as follows.

In the following section the high-energy limit of SE for polynomial potentials is considered and its standard form is established.

In section 3 Stokes graphs (SG) for the standardized high-energy polynomial potential are considered and their possible standard forms are established in this limit.

In the following section we define the full set of fundamental solutions for the standard high energy limit potential $(-i\alpha z)^n$ and formulate the basic lemma on possible positions of roots of the FS's.

In section 5 two theorems are formulated establishing the precise positions of roots of FS's in the high energy limit.

In section 6 the results of the previous section are extrapolated to an arbitrary polynomial potential rescaled correspondingly to the limit considered.

In section 7 the quantized cases of FS's and an influence of the quantization on distributions of zeros of FS's are investigated.

We conclude and summarize the results in section 8.

2. High energy limit of Stokes graphs for polynomial potentials

Consider the stationary 1D Schrödinger equation with a polynomial potential $P_n(z) = a_n z^n + \cdots + a_1 z$ where $a_n \neq 0, n > 1$, and all $a_i, i = 1, \ldots, n$, are, in general, complex. A free term of $P_n(z)$ is assumed to be absorbed by the complex energy parameter E while the remaining ones are assumed to be energy independent. We have

$$\phi''(z) - \frac{2m}{\hbar^2}(P_n(z) - E)\phi(z) = 0. \tag{1}$$

To standardize our problem we make the substitution $z \to -i\alpha\left(\frac{E}{a_n}\right)^{\frac{1}{n}} z$ with $\alpha = 1$ for n odd and $\alpha = 1$ or $e^{-i\frac{\pi}{n}}$ for n even so that we get for $\psi_\alpha(z) \equiv \phi\left(-i\alpha\left(\frac{E}{a_n}\right)^{\frac{1}{n}} z\right)$

$$\psi_\alpha''(z) - \lambda^2 W_n(z, \lambda)\psi_\alpha(z) = 0 \tag{2}$$

where

$$\begin{gathered} W_n(z,\lambda) \equiv (-i\alpha z)^n - 1 + \sum_{k=1}^{n-1} b_{n-k}(-i\alpha)^{\frac{2k}{n+2}}\lambda^{-\frac{2k}{n+2}}(-i\alpha z)^{n-k} \\ \lambda^2 = -\alpha^2\frac{2m}{\hbar^2}a_n^{-\frac{2}{n}}E^{\frac{n+2}{n}}, \qquad b_{n-k} = \frac{a_{n-k}}{a_n^{\frac{n-k+2}{2(n+2)}}}\left(\frac{2m}{\hbar^2}\right)^{\frac{k}{n+2}}, \qquad k = 1, \ldots, n-1, \end{gathered} \tag{3}$$

where the factor $-i\alpha$ has been introduced for a convenience that will be self-explaining later.

Let us define also by

$$W_n^\infty(z) \equiv (-i\alpha z)^n - 1 = \lim_{|\lambda|\to\infty} W_n(z, \lambda) \tag{4}$$

the limit form approached by $W_n(z, \lambda)$ when $|E| \to \infty$.

Let us remind further that for the case considered the Stokes graph (SG) is created as the set of all lines (Stokes lines (SL)) emerging from each root $z_i(\lambda), i = 1, \ldots, n$, of $W_n(z, \lambda)$ and satisfying one of the following equations:

$$\mathrm{Re}\left(\lambda\int_{z_i(\lambda)}^{z}\sqrt{W_n(\xi,\lambda)}\,d\xi\right) = 0, \qquad i = 1, \ldots, n \tag{5}$$

The roots $z_i(\lambda), i = 1, \ldots, n$ will also be called turning points.

From (3) and (4) it is obvious that for $|E| \to \infty$, i.e. $|\lambda| \to \infty$, the limit Stokes graph corresponding to the rescaled problem is determined by the roots (turning points) of $W_n^\infty(z)$ and the phase of λ. It is also clear that for the considered limit the roots $z_i(\lambda)$ are all simple and their loci are close to those of $W_n^\infty(z)$ so that the SG corresponding to $W_n(z,\lambda)$ differs only slightly from that corresponding to $W_n^\infty(z)$ and can be obtained from the latter SG by its small deformation.

By the above definition the Stokes lines and the corresponding Stokes graph are defined in the considered limit on the two sheeted Riemann surface R_2 with the turning points of $W_n^\infty(z)$ as the branch points of this surface. However since on these two sheets the values of $\sqrt{W_n^\infty(z)}$ differ by a sign only the projections on the z-plane of the Stokes lines defined on each sheet coincide.

Therefore considering a pattern of SL's on the cut z-plane $C_{\rm cut}$ with cuts emerging from the turning points of $W_n^\infty(z)$ we see that the SL's on $C_{\rm cut}$ are quasi continuous on the cuts despite the fact that they are pieces of different SL's collected from the two sheets of R_2.

In general, a number of SL's emerging from a given turning point $z_i(\lambda)$ depends on its order. In the simplest case of the simple turning point there are exactly three SL's emerging from it. Each of them can run to infinity of $C_{\rm cut}$ or end at another turning point $z_j(\lambda)$. A SL with the last property is called the *inner* one.

A SG is called *critical* if at least one of its SL's is the inner one. It is called *not critical* in the opposite case.

It is worth noting that a variety of the non-critical SG's for a given potential is much richer than the critical ones what can be seen already at the level of the general complex harmonic potential $P_2(z) = a(z-b)(z-c)$ with a, b, c to be complex numbers. In this case the critical SG is defined by the condition $\mathrm{Re}\int_b^c \sqrt{a(z-b)(z-c)}\,\mathrm{d}z = \mathrm{Re}(\mathrm{i}\pi a^{\frac{1}{2}}(c-b)^2) = \mathrm{Im}(\pi a^{\frac{1}{2}}(c-b)^2) = 0$, i.e. by one real algebraic equation. Therefore the variety of the critical SG's in this case is equivalent to the two five-dimensional real manifolds embedded in the six-dimensional one of all SG's corresponding to the case. The potentials with $a > 0, b, c$ real provide us therefore with critical SG's while for $a < 0, b, c$ real the corresponding SG's are non-critical.

3. Properties of Stokes graphs corresponding to the potentials $(-\mathrm{i}\alpha z)^n$

Consider now the SG's corresponding to $W_n^\infty(z)$. For this goal assume λ to be real and positive for a while.

For $\alpha = 1$ there is a root of $W_n^\infty(z)$ at $z = z_0 = \mathrm{i}$ while the remaining ones are regularly distributed on the circle $|z| = 1$ being located at $z = z_k = \mathrm{i}\,\mathrm{e}^{\mathrm{i}\frac{2k\pi}{n}}, k = \pm 1, \pm 2, \ldots, \pm[\frac{n-1}{2}]$ and at $z = z_{\frac{n}{2}} = -\mathrm{i}$ for even n, so that the corresponding pairs of them satisfy the relation $z_k = -\bar{z}_{-k}$ (where bar over z means its complex conjugation), i.e. these pairs are located symmetrically with respect to the imaginary axis.

For even n and $\alpha = \mathrm{e}^{-\mathrm{i}\frac{\pi}{n}}$ all zeros are given by pairs $z_{\pm k} = \mathrm{i}\,\mathrm{e}^{\pm\mathrm{i}\frac{(2k+1)\pi}{n}}, k = 1, 2, \ldots, \frac{n}{2}$ satisfying again the relations $z_k = -\bar{z}_{-k}$.

Let us denote by $A_{n,\alpha}$ the full set of indices just established enumerating the turning points z_l, i.e. $l \in A_{n,\alpha}$ for each z_l and corresponding to given n and α and by $Z_{n,\alpha}$ the sets of the corresponding roots themselves.

By the definitions of the sets $Z_{n,\alpha}$ we have the following simple relations between them

$$\begin{aligned}
&\mathrm{e}^{\pm\mathrm{i}\frac{2\pi}{n}} Z_{n,\alpha} = Z_{n,\alpha} && \text{for all } n \text{ and } \alpha\\
&\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{n}} Z_{n,1} = -Z_{n,1} && \text{for odd } n\\
&\mathrm{e}^{\pm i\frac{\pi}{n}} Z_{n,\mathrm{e}^{-\mathrm{i}\frac{\pi}{n}}} = Z_{n,1} && \text{for even } n\\
&\mathrm{e}^{\pm i\frac{\pi}{n}} Z_{n,1} = Z_{n,\mathrm{e}^{-\mathrm{i}\frac{\pi}{n}}} && \text{for even } n
\end{aligned} \tag{6}$$

where $-Z_{n,1}$ means the set of all $-z_l, z_l \in Z_{n,1}$.

It can be shown by a direct calculation (see appendix A and [6]) that

$$\int_{z_k}^{z_{-k}} \sqrt{(-\mathrm{i}\alpha z)^n - 1}\,\mathrm{d}z = \begin{cases} \dfrac{2\mathrm{i}}{n}\sin\dfrac{2k\pi}{n}B\left(\dfrac{3}{2},\dfrac{1}{n}\right) & \text{for} \quad \alpha = 1 \\ \dfrac{2\mathrm{i}}{n}\sin\dfrac{(2k-1)\pi}{n}B\left(\dfrac{3}{2},\dfrac{1}{n}\right) & \text{for} \quad \alpha = \mathrm{e}^{-\frac{\mathrm{i}\pi}{n}} \end{cases} \qquad k \in A_{n,\alpha} \tag{7}$$

where $B(x, y)$ is the beta function.

Therefore

$$\mathrm{Re}\int_{z_k}^{z_{-k}} \sqrt{W_n^\infty(z)}\,\mathrm{d}z = 0, \qquad k = 1, 2 \ldots, \tag{8}$$

for any such a pair, i.e. each pair z_k, z_{-k} lies on the same Stokes line.

Let us further note that the following relation holds:

$$\int_{z_k}^{z_{-k}} \sqrt{(-\mathrm{i}\alpha z)^n - 1}\,\mathrm{d}z = \mathrm{e}^{\pm\mathrm{i}\frac{\pi}{n}} \int_{z_{k'}}^{z_{-k'\pm1}} \sqrt{(-\mathrm{i}\alpha' z)^n - 1}\,\mathrm{d}z, \tag{9}$$

where α' are the corresponding α's on the rhs in (6) for the corresponding α's on the lhs of (6) and $z_{k'}, z_{-k'\pm1}$ are in the corresponding sets on the rhs of (6), i.e. $z_{k'}, z_{-k'\pm1} \in Z_{n,\alpha'}$ if $z_k, z_{-k} \in Z_{n,\alpha}$.

The relation (9) means that according to the definition of SL's by (5) there is the inner SL between the turning points $z_k, z_{-k\pm1}$ of the set $Z_{n,\alpha}$ if $\arg\lambda = \mp\frac{\pi}{n}$, i.e.

$$\mathrm{Re}\left(|\lambda|\,\mathrm{e}^{\pm\frac{\mathrm{i}\pi}{n}} \int_{z_k}^{z_{-k\mp1}} \sqrt{(-\mathrm{i}\alpha z)^n - 1}\,\mathrm{d}z\right) = 0 \tag{10}$$

Let us remind further that from each root of $W_n^\infty(z)$ (all the roots are simple) emerge three SL's. If these roots are members of a pair $z_k, z_{-k}, k \in A_{n,\alpha}$, then one of these lines runs from one root to another while the remaining pairs of SL's run to infinity of the z-plane.

If a root of $W_n^\infty(z)$ is not a member of any pair of them then all three SL's which emerge of it run to the infinity of the z-plane.

Each pair of neighbour SL's emerging from the same root and running to the infinity forms a *sector* while the SL's alone lie on its boundary.

However for an odd n there is still a sector whose boundary is formed by the neighbouring SL's running to infinity and emerging from the last pair (with the highest value of $k = \frac{n-1}{2}$) of roots and by the SL linking this pair.

It is easy to note that $2n + 4$ is the total number of sectors lying on R_2. However because of coincidence of SL's projected on C_{cut} we can consider on this cut z-plane quasi sectors formed by these projected SL's. There are now $n + 2$ of such quasi sectors which will be called again sectors for simplicity. We can enumerate them correspondingly to roots attached to their boundaries.

So for each pair of turning points $z_k, z_{-k}, k \in A_{n,\alpha}$, the corresponding sectors are denoted by S_k and S_{-k} respectively while the single sector formed by the last pair of roots in the odd-n case is denoted by $S_{\frac{n+1}{2}}$ and by $S_{\frac{n+2}{2}}$ in the even-n case and in the same case the single sector formed by the first pair of roots is denoted by S_0.

If there are single roots, at $z = z_0 = \mathrm{i}$ or $z = z_{\frac{n}{2}} = -\mathrm{i}$, then the two sectors connected with the first root we denote by $S_{\frac{n+3}{2}}$ (the left one) and $S_{-\frac{n+3}{2}}$ for n odd and by $S_{\frac{n+2}{2}}$ (the left one) and $S_{-\frac{n+2}{2}}$ for n even. The second root at $z_{\frac{n}{2}}$ can exist only for n even and the two sectors connected with it are denoted by $S_{\frac{n}{2}}$ (the left) and $S_{-\frac{n}{2}}$.

Let us denote by $B_{n,\alpha}$ the full set of indices used to enumerate all the sectors.

Therefore typical SG's for the cases considered look as in figures 1(*a*)–(*e*).

Note that on these figures (as well as on the remaining ones) only topological relations between SL's are represented correctly while the corresponding metric ones only roughly. In particular one has to have in one's mind that all SL's emerging to infinity on the figures 1(*a*)–(*e*) (as well as on all the remaining ones in this paper containing SG's) are collected into bunches so that all SL's in a given bunch have the common asymptote which is determined by a ray emerging from $z = 0$ called Stokes ray. There are exactly $n + 2$ of such bunches and the same number of the corresponding Stokes rays for the polynomial $P_n(z)$, the rays being displayed by equal angles on C_{cut}. In the case of figures 1(*a*)–(*e*) each bunch of SL's can contain at most two such lines.

Let us note further that incorporating the senior coefficient a_n of $P_n(z)$ into the definition of the parameter λ has allowed us to avoid considering non-standard distributions of the limit loci of roots of $W_n(z,\lambda)$, i.e. rotated by some angle with respect to the standard ones which would be the case if for example λ was defined by the absolute value of a_n only.

However such a standardization of the positions of roots has standardized also to some extent possible patterns of Stokes lines namely by making them sensitive to a change of $\arg\lambda$ only.

Consider now again the SG's corresponding to the standard configurations of roots for real λ. They are shown in figures 1(*a*)–(*e*). What happens when λ acquires a nonvanishing phase β? It is well known that in such a case each three SL's emerging from the same root rotate by the angle $-\frac{2\beta}{3}$ around the root while the Stokes rays of the corresponding SG rotate by $\frac{2\beta}{n+2}$ around the infinity point. Of course for $\beta = \pm\pi$ the whole SG comes back to its form for real λ.

However patterns of SG's arising from this β-rotations are not all topologically different because of symmetric distributions of turning points. Namely, if we consider the SL linking a pair z_k, z_{-k} (call it L_k) and start to rotate the SG by sufficiently small $\beta > 0$ then, since all SL's of the graph will rotate anticlockwise, L_k will split into two SL's now running to the infinity each. Still enlarging β we can cause one of these splitting SL's emerging from z_k to meet the root z_{-k-1}, i.e. the two last roots find themselves on the same SL. But as it follows from (10) this can happen only when $\beta = \frac{\pi}{n}$ and from (9) that this rotation of λ can be compensated by the opposite rotation of the C-plane which brings the rotated SG again to its standard form of figures 1(*a*)–(*e*) for $\arg\lambda = 0$.

Therefore as it follows from the above discussion it is enough to rotate the standard SG by angles β chosen from the interval $\left(-\frac{\pi}{n}, \frac{\pi}{n}\right)$ to handle all topologically nonequivalent configurations of SG for the standard distributions of roots.

Let us put therefore $\lambda = |\lambda|\mathrm{e}^{\mathrm{i}\beta}$, $0 < |\beta| < \frac{\pi}{n}$, in the condition

$$\mathrm{Re}\left(\lambda\int_{z_k}^{z}\sqrt{W_n^{\infty}(\xi)}\,\mathrm{d}\xi\right) = 0 \tag{11}$$

for each z_k, $k \in A_{n,\alpha}$.

It then follows from the above discussion that all SG's corresponding to the λ's chosen are non-critical, i.e. each of their SL's defined by (11) runs to the infinity of the z-plane, see figure 2.

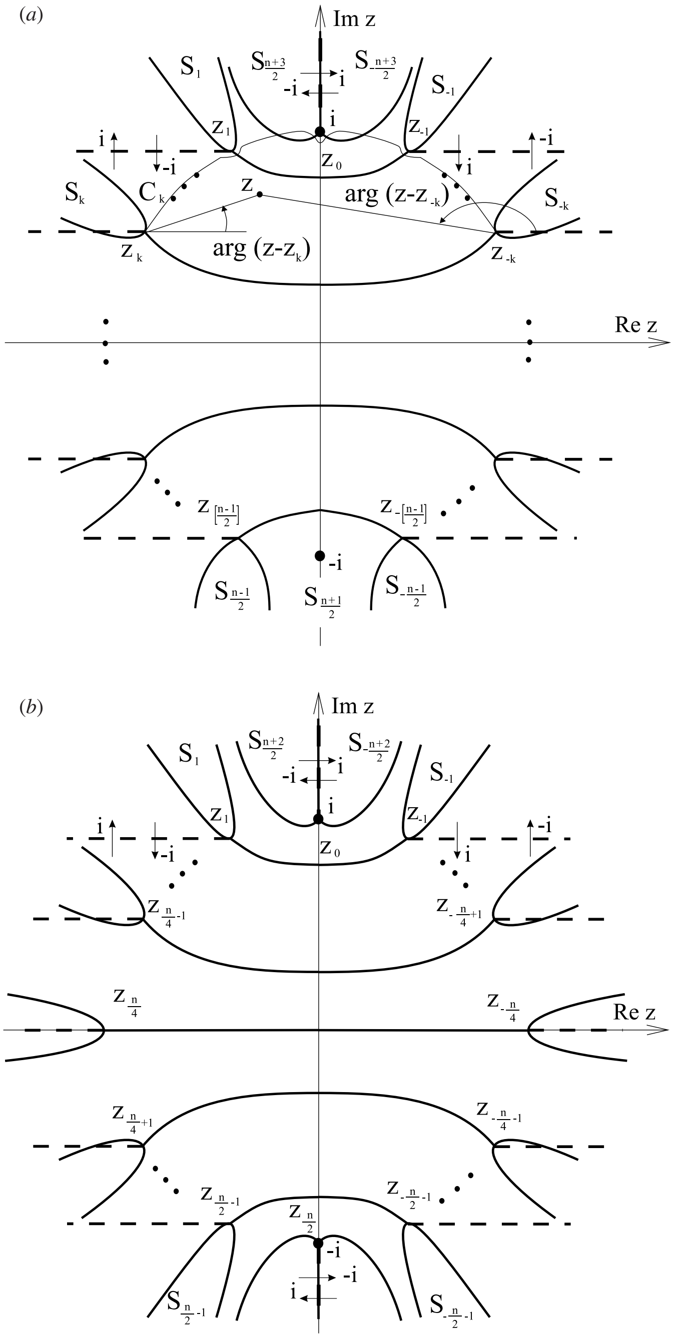

Figure 1. (*a*) SG for an odd n. Broken lines are cuts and the semicircle C_k linking the points z_k and z_{-k} (which is not a SL) is the integration contour in the formula (7). (*b*) SG for an even n and $\alpha = 1$ with odd number of internal SL's. Broken lines are cuts. (*c*) SG for an even n and $\alpha = 1$ with even number of internal SL's. (*d*) SG for an even n and $\alpha = \mathrm{e}^{-\frac{i\pi}{n}}$ with odd number of internal SL's. (e) SG for an even n and $\alpha = \mathrm{e}^{-\frac{i\pi}{n}}$ with even number of internal SL's.

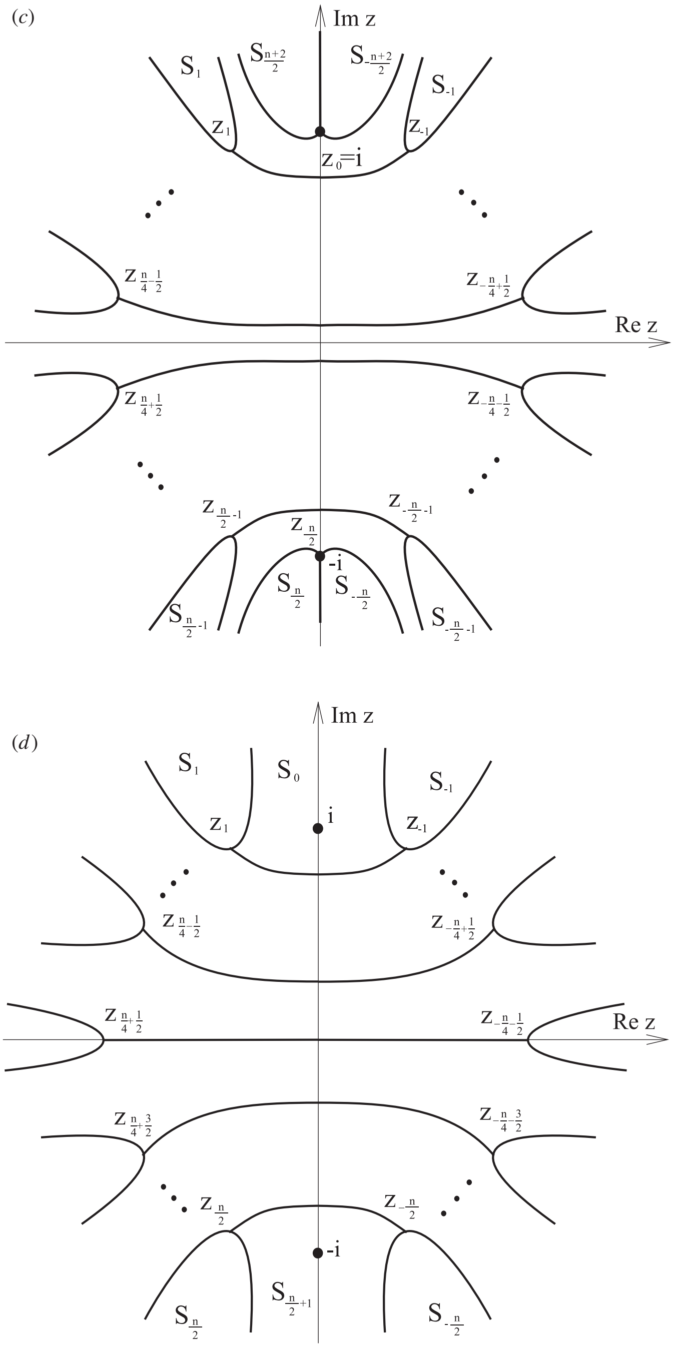

Figure 1. (Continued.)

We shall show in the following sections that for the potentials $(-\mathrm{i}\alpha z)^n$ and $W_n(z) + 1$ in the limit $|\lambda| \to \infty$ all zeros of each of their fundamental solutions are distributed along a boundary of its canonical domain, while this boundary is a collection of Stokes lines.

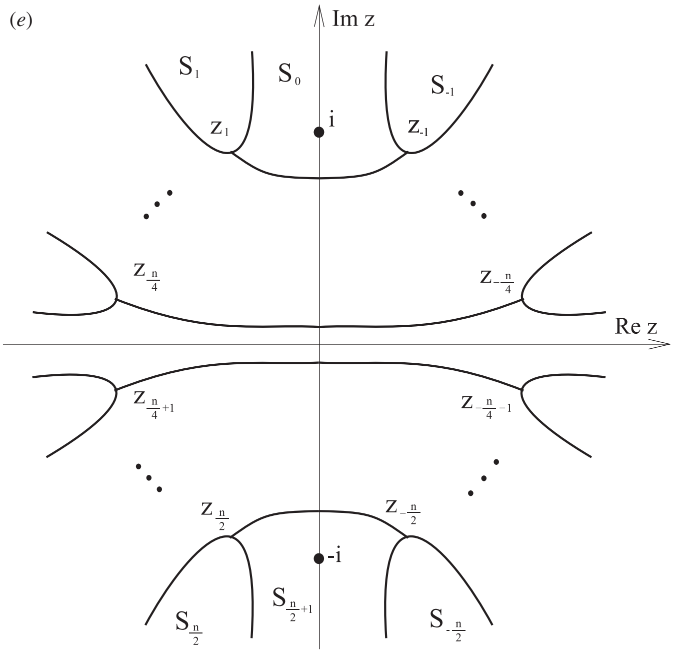

Figure 1. (Continued.)

4. Fundamental solutions for the potential $(-\mathrm{i}\alpha z)^n$

The fundamental solutions (FS) of the equation

$$\psi''(z) - \lambda^2((-\mathrm{i}\alpha z)^n - 1)\psi(z) = 0 \tag{12}$$

are solutions defined on R_2 separately in each of $2n+4$ sectors. They are subdominant in the sectors which they are defined in, i.e. they vanish for $|z| \to \infty$ inside the sectors. They can be given explicit forms of the following functional series [3]

$$\psi_k^\infty(z,\lambda) = \left(W_n^\infty(z)\right)^{-\frac{1}{4}} \mathrm{e}^{\sigma_k \lambda \tilde{W}_n^\infty(z,z_k)} \chi_k^\infty(z,\lambda) \tag{13}$$

where $z \in S_k$ and z_k is a turning point lying on the boundary of S_k while

$$\begin{aligned} \tilde{W}_n^\infty(z,z_k) &= \int_{z_k}^{z} \sqrt{W_n^\infty(y)}\,\mathrm{d}y \\ \chi_k^\infty(z,\lambda) &= 1 + \sum_{p\geqslant 1} \left(-\frac{\sigma_k}{2\lambda}\right)^n Y_{k;p}^\infty(z,\lambda) \end{aligned} \tag{14}$$

and

$$\begin{aligned} Y_{k;p}^\infty(z,\lambda) = \int_{\gamma_k(z)} \mathrm{d}y_1 \int_{\gamma_k(y_1)} \mathrm{d}y_2 \cdots \int_{\gamma_k(y_{p-1})} \mathrm{d}y_p \omega_\infty(y_1)\omega_\infty(y_2)\cdots\omega_\infty(y_p) \\ \times\left(1 - \mathrm{e}^{-2\sigma_k\lambda\tilde{W}_n^\infty(z,y_1)}\right)\left(1 - \mathrm{e}^{-2\sigma_k\lambda\tilde{W}_n^\infty(y_1,y_2)}\right)\cdots\left(1 - \mathrm{e}^{-2\sigma_k\lambda\tilde{W}_n^\infty(y_{p-1},y_p)}\right) \\ n \geqslant 1 \end{aligned} \tag{15}$$

with

$$\omega_\infty(z) = \frac{5}{16}\frac{\left(W_n^\infty(z)\right)'^2}{\left(W_n^\infty(z)\right)^{\frac{5}{2}}} - \frac{1}{4}\frac{\left(W_n^\infty(z)\right)''}{\left(W_n^\infty(z)\right)^{\frac{3}{2}}} = \left(W_n^\infty(z)\right)^{-\frac{1}{4}}\left(\left(W_n^\infty(z)\right)^{-\frac{1}{4}}\right)''. \tag{16}$$

The signatures $\sigma_k = \pm 1$ present in formulae (13)–(15) are defined in each particular sector S_k in such a way to ensure the inequality $\mathrm{Re}\left(\sigma_k \lambda \tilde{W}_n^{\infty}(z, z_k)\right) < 0$ to be satisfied in this sector.

The integration paths $\gamma_k(z)$ in (15) which start from the infinities of the corresponding sectors are chosen in such a way to satisfy the inequality $\mathrm{Re}\left(\sigma_k \lambda \tilde{W}_n^{\infty}(y_i, y_{i+1})\right) \geqslant 0$, $y_i, y_{i+1} \in \gamma_k(z)$, for each factor of the integrations in (15) so that $\left|1 - \mathrm{e}^{-2\sigma_k \lambda \tilde{W}_n^{\infty}(y_i, y_{i+1})}\right| \leqslant 2$ on $\gamma_k(z)$ and this together with $|\omega_\infty(z)| \sim |z|^{-\frac{1}{2}n-2}$ for $|z| \to \infty$ ensures the absolute convergence of the multiple integral in (15) along $\gamma_k(z)$. Such paths are called canonical and a point z for which $\gamma_k(z)$ can be chosen to be canonical is also called canonical with respect to the sector S_k. The latter property is completely of a topological nature. It is easy to see that each point of the sector S_k is its canonical point.

It follows also from the above definition of canonical points that the series (14) defining the solution $\psi_k^{\infty}(z, \lambda)$ is convergent in each canonical point and therefore in the sector S_k. In fact the domain of its convergence is larger than the sector S_k and the convergence of the series inside this domain is uniform (see lemma below).

According to our earlier discussion we can assume the argument of λ to vary in the interval $\left(-\frac{\pi}{n}, +\frac{\pi}{n}\right)$.

Let us note however that there is no necessity to define the fundamental solutions and consider them on the whole R_2. It is enough to do it on C_{cut} only because of the following reasons.

As we have mentioned earlier there is one-to-one correspondence between the sectors lying on the C_{cut} which can be considered as a one sheet of R_2 and the sectors lying on the second sheet C'_{cut} of R_2 the latter sheet being then a complement of C_{cut} to R_2 connected with C_{cut} by cuts. The correspondence between two such sectors is built by their coincidence when C'_{cut} is projected on C_{cut}.

Let $S \subset C_{\mathrm{cut}}$ and $S' \subset C'_{\mathrm{cut}}$ be a pair of such sectors. Then the two fundamental solutions defined in each of these two sectors coincide up to a constant. This coincidence is visible in the form given by (13)–(16) by the invariance of $\sigma_k \tilde{W}_n^{\infty}(z, z_k)$ and $\sigma_k \omega_\infty(z)$ when passing from the sector S to S' since then σ_k, $\tilde{W}_n^{\infty}(z, z_k)$ and $\omega_\infty(z)$ change their signs simultaneously. Only the common factor $\left(W_n^{\infty}(z)\right)^{-\frac{1}{4}}$ of the solutions changes slightly under this operation acquiring one of the phase factors $\pm\mathrm{i}$.

Therefore we can consider the fundamental solutions defined only on C_{cut} and the solution $\psi_k^{\infty}(z, \lambda)$ given by (13)–(16) is defined in the sector S_k the latter being enumerated in the way described earlier. However if $\psi_k^{\infty}(z, \lambda)$ is defined in the sector crossed by a cut of C_{cut} we have to remember about necessary changes in the forms (13)–(16) of the solution described above to keep its identity when the cut is crossed. This latter note is valid also when such a cut is crossed by any of the fundamental solutions if the latter is continued analytically along a canonical path.

A collection $D_k \subset C_{\mathrm{cut}}$ of all points canonical with respect to S_k is called a canonical domain corresponding to S_k. Of course $S_k \subset D_k$ for each k [3].

A boundary od D_k is composed of some Stokes lines.

A general rule for a given SL to belong to ∂D_k is that a canonical path $\gamma_k(z)$ when z approaches the line has to cross another SL emerging from the same turning point.

For non-critical SG's (i.e. all SL's of which run to the infinities) ∂D_k is composed of all turning points and of single SL's emerging from these turning points, testing according the rule just mentioned, see figure 2.

In other cases (i.e. for critical positions of SG) all three SL's emerging from a turning point can belong to ∂D_k. The latter point is then just the root z_{-k} joined by one of its Stokes

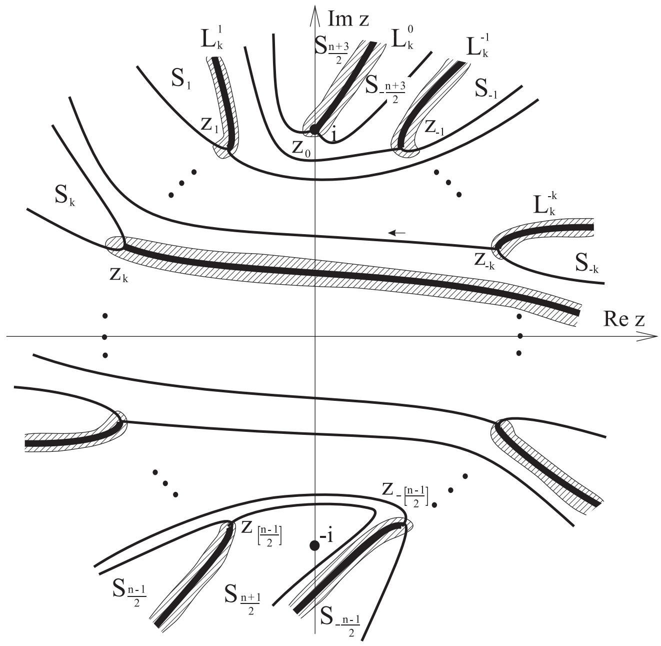

Figure 2. Exceptional SL's and their $V_{k,\epsilon}$-vicinity for $\psi_k^\infty(z,\lambda)$. The non-critical case ($\arg\lambda \neq 0$).

lines with the turning point z_k, i.e. by the line which start from the former point and end at the latter. Such a line is then called an *inner* one. All points of the sector S_{-k} cannot be then joined with the infinity of S_k by cannonical paths and therefore do not belong to D_k.

Each ∂D_k is specific for its canonical domain D_k.

In what follows we will consider also a $C_{\text{cut}}(\epsilon)$-plane arising from C_{cut} by depriving the latter ϵ-vicinities of the turning points z_k, $k \in A_{n,\alpha}$. Namely, for any $\epsilon > 0$ denote by $\Delta_k(\epsilon)$ the following circle vicinities of turning points z_k: $\Delta_k(\epsilon) = \{z : |z - z_k| < \epsilon\}$, $k \in A_{n,\alpha}$. Then $C_{\text{cut}}(\epsilon) = C_{\text{cut}} \setminus \bigcup_{k\in A_{n,\alpha}} \bar{\Delta}_k(\epsilon)$.

Let L_k^r denote the connected set of SL's contained in ∂D_k and emerging from the turning point z_r. Therefore for the non-critical SG's each L_k^r is then a single SL's and different L_k^r's are disjoint pairways while for the critical SG's a unique difference with the previous case is connected with L_k^k and L_k^{-k} which are not disjoint, the first one containing the inner SL between z_k and z_{-k} while the second one containing all the three SL's emerging from z_{-k}, therefore also the inner SL between z_k and z_{-k}.

The lines L_k^r, $r \in A_{n,\alpha}$, will be called (after Eremenko *et al* [9]) *exceptional* with respect to the solution $\psi_k^\infty(z,\lambda)$. Of course they are exceptional in that none of their points can be reached by the solution $\psi_k^\infty(z,\lambda)$ if the latter is to be continued to them along canonical paths.

In fact a full set of exceptional SL's characterizes entirely the boundary ∂D_k of the canonical domain D_k by the equation: $\bigcup_{r\in A_{n,\alpha}} L_k^r = \partial D_k$.

For a given L_k^r let us denote by $V_k^r(\epsilon)$ an ϵ-vicinity of this set of SL's defined by the following conditions:

(1) $L_k^r \subset V_k^r(\epsilon)$.

(2) the boundary of $V_k^r(\epsilon)$ consists of (at most two) continuous lines an Euclidean distance of which to L_k^r is smaller than ϵ (see figures 2 and 3).

Let us further denote by $D_{k,\epsilon}$ a subset of D_k given by $D_{k,\epsilon} = D_k \setminus \bar{V}_k(\epsilon)$ where $V_k(\epsilon) = \bigcup_{r\in A_{n,\alpha}} V_k^r(\epsilon)$.

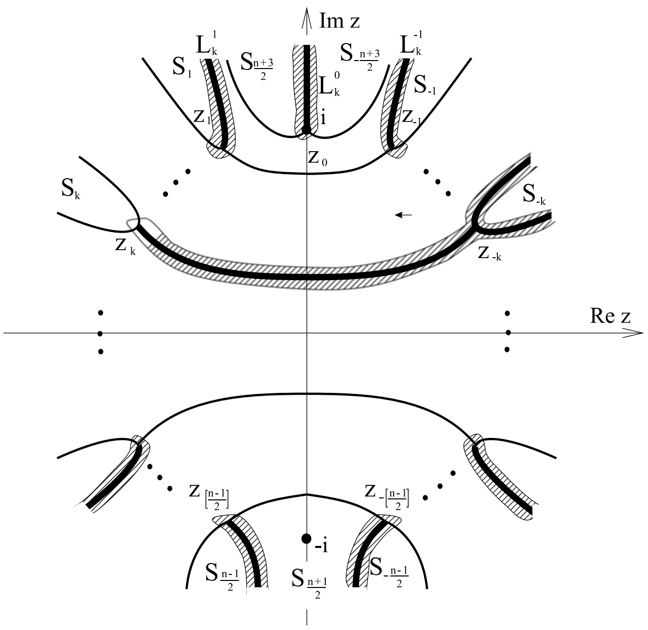

Figure 3. Exceptional SL's and their $V_{k,\epsilon}$-vicinity for $\psi_k^\infty(z,\lambda)$. The critical case ($\arg\lambda = 0$).

It is clear that $V_k(\epsilon)$ is an ϵ-vicinity of ∂D_k (see figures 2 and 3.)

Needless to say for the non-critical SG's every turning point z_l belongs to $V_k(\epsilon)$ together with exactly one of its three SL's which emerge from it. In the case of the critical SG's the same is true for all z_l excluding z_{-k} which belong to $V_k(\epsilon)$ together with its three SL's if z_k is connected with z_{-k} by the inner SL.

The following property of FS's is the key one for our further considerations.

Lemma. *In the domain $D_{k,\epsilon}$ the factor $\chi_k^\infty(z,\lambda)$ of the solution (13) satisfies the following bound*

$$
\begin{aligned}
&|\chi_k^\infty(z,\lambda) - 1| \leqslant \mathrm{e}^{\frac{C_\epsilon^\infty}{\lambda_0}} - 1, \qquad |\lambda| > \lambda_0 \\
&C_\epsilon^\infty = \liminf_{\gamma_k(z), z\in D_{k,\epsilon}, k\in A_{n,\alpha}} \int_{\gamma_k(z)} |\omega_\infty(\xi)\,\mathrm{d}\xi| < \infty
\end{aligned}
\tag{17}
$$

where $\gamma_k(z)$ are canonical.

We have left the proof of lemma to appendix B.

There is a direct relation between above lemma and the so-called semiclassical expansions of χ^∞-factors of FS's (13) for $\lambda \to \infty$ but fixed z which will be needed in our further considerations. Such expansions have been considered in our earlier papers (see [4], ref. 1 and [5], ref. 5) and have been given the following exponential forms:

$$
\begin{aligned}
\chi_k^\infty(z,\lambda) \sim \chi_k^{\infty(as)}(z,\lambda) &= 1 + \sum_{p\geqslant 1}\left(-\frac{\sigma_k}{2\lambda}\right)^p \tilde{Y}_{k;p}^\infty(z) = \sum_{p\geqslant 0}\left(-\frac{\sigma_k}{2\lambda}\right)^p \tilde{Y}_{k;p}^\infty(z) \\
&= \exp\left(\sum_{p\geqslant 1}\left(-\frac{\sigma_k}{2\lambda}\right)^p \int_{\infty_k}^z X_p^\infty(y)\,\mathrm{d}y\right)
\end{aligned}
\tag{18}
$$

where

$$\tilde{Y}^{\infty}_{k;0}(z) \equiv 1$$

$$\tilde{Y}^{\infty}_{k;p}(z) = \int_{\infty_k}^{z} \mathrm{d}y_n \left(W_n^{\infty}(y_p)\right)^{-\frac{1}{4}} \left(\left(W_n^{\infty}(y_n)\right)^{-\frac{1}{4}} \int_{\infty_k}^{y_p} \mathrm{d}y_{p-1} \left(W_n^{\infty}(y_{p-1})\right)^{-\frac{1}{4}} \right.$$
$$\left. \times \left(\cdots \left(W_n^{\infty}(y_2)\right)^{-\frac{1}{4}} \int_{\infty_k}^{y_2} \mathrm{d}y_1 \left(W_n^{\infty}(y_1)\right)^{-\frac{1}{4}} \left(\left(W_n^{\infty}(y_1)\right)^{-\frac{1}{4}}\right)'' \cdots \right)''\right)'', \quad p \geqslant 1 \tag{19}$$

and

$$\sum_{p\geqslant 1} \left(-\frac{\sigma_k}{2\lambda}\right)^p X_p^{\infty}(z) \equiv Z_k^{\infty}(z,\lambda) = \frac{1}{\chi_k^{\infty(as)}(z,\lambda)} \frac{\mathrm{d}\chi_k^{\infty(as)}(z,\lambda)}{\mathrm{d}z} \tag{20}$$

Note that after the first operation in (19) (i.e. after the integration of $\omega_\infty(y_1)$ followed by multiplication by $\left(W_n^{\infty}(y_1)\right)^{-\frac{1}{4}}$ and the double differentiation of the final result) one gets a function which vanishes as $y_1^{-\frac{3}{4}n-2}$ when $y_1 \to \infty$ while each next such operation lowers the power of the integrated variable at infinity by $-\frac{1}{4}n - 1$. Therefore $\tilde{Y}^{\infty}_{k;p}(z)$, $p \geqslant 1$, are all well defined.

Note also that $X_p^{\infty}(y)$, $p \geqslant 1$, are sector independent holomorphic point functions on the C_{cut}-plane which are given by the following recurrent formula (see [4], ref. 2):

$$X_1^{\infty}(z) = \omega_\infty(z) = \left(W_n^{\infty}(z)\right)^{-\frac{1}{4}} \left(\left(W_n^{\infty}(z)\right)^{-\frac{1}{4}}\right)'' = \left(W_n^{\infty}\right)^{-\frac{1}{2}} \frac{U_{2n-2}(z)}{\left(W_n^{\infty}(z)\right)^2}$$

$$X_p^{\infty}(z) = -\frac{1}{2}\left(W_n^{\infty}(z)\right)^{-\frac{3}{2}} \left(W_n^{\infty}(z)\right)' X_{p-1}^{\infty}(z)$$
$$+ \left(W_n^{\infty}(z)\right)^{-\frac{1}{2}} (z) \left(\sum_{k=1}^{m-2} X_k^{\infty} X_{m-k-1}^{\infty} + \left(X_{p-1}^{\infty}(z)\right)'\right), \qquad p = 2, 3, \ldots \tag{21}$$

where $U_{2n-2}(z)$ is a polynomial of the $2n-2$-degree.

It follows from (21) that X_{2m}^{∞}, $m \geqslant 1$, have only poles at the turning points while X_{2m+1}^{∞}, $m \geqslant 0$, have there the square root branch points. Therefore the same are the properties of $Z_\infty^{+}(z,\lambda)$ and $Z_\infty^{-}(z,\lambda)$ at these points respectively where $Z_\infty^{+}(z,\lambda) + \sigma_k Z_\infty^{-}(z,\lambda) = Z_k^{\infty}(z,\lambda)$ and

$$Z_\infty^{+}(z,\lambda) = \sum_{m\geqslant 1} \left(\frac{1}{2\lambda}\right)^{2m} X_{2m}^{\infty}(z)$$
$$Z_\infty^{-}(z,\lambda) = \sum_{m\geqslant 0} \left(\frac{1}{2\lambda}\right)^{2m+1} X_{2m+1}^{\infty}(z). \tag{22}$$

The series in (22) are asymptotic, i.e. they are in general divergent. Therefore the analytic properties of $Z_\infty^{\pm}(z,\lambda)$ and $Z_k^{\infty}(z,\lambda)$ on the C_{cut}-plane can be considered only formally as a known collection of singularities of all $X_m^{\infty}(z)$, $m \geqslant 1$.

If we now take into account that $\chi_{i\to j}^{\infty}(\lambda) \equiv \lim_{z\to\infty_j} \chi_i^{\infty}(z,\lambda) = \chi_{j\to i}^{\infty}(\lambda)$ where ∞_j is the infinite point of the sector S_j communicated canonically with the sector S_i (see ref. 1 of [4] and ref. 5 of [5]) then we get

$$\mathrm{e}^{\int_{\infty_i}^{\infty_j}(Z_\infty^{+}+\sigma_i Z_\infty^{-})\,\mathrm{d}z} = \mathrm{e}^{\int_{\infty_j}^{\infty_i}(Z_\infty^{+}+\sigma_j Z_\infty^{-})\,\mathrm{d}z}. \tag{23}$$

Since however $\sigma_i = -\sigma_j$ then we get from (23)

$$\int_{\infty_i}^{\infty_j} Z_\infty^+(z,\lambda)\,\mathrm{d}z = 0 \tag{24}$$

or

$$\int_{\infty_i}^{\infty_j} X_{2m}^\infty(z)\,\mathrm{d}z = 0, \qquad m \geqslant 1 \tag{25}$$

for any pair of canonically communicated sectors.

However the integration in (25) is now not limited by canonical paths since under the integral there are now no exponentials limiting this integration to canonical paths and all $X_{2m}^\infty(z)$, $m \geqslant 1$ are holomorphic point function on the C_{cut}-plane, i.e. these paths can be freely deformed with the integral being still convergent. It is easy to see that because of that a given integral (25) can be deformed to any other integral between any pair of sectors (i.e. not necessarily communicated canonically) as well as to any integral along an arbitrary loop. It means that residua of $X_{2m}^\infty(z)$, $m \geqslant 1$ and therefore also of $Z_\infty^+(z,\lambda)$ at the poles which they have at the turning points have to vanish.

Therefore we conclude that the Riemann surface of $Z_\infty^+(z,\lambda)$ is just the C-plane on which it is formally meromorphic with vanishing residua at its poles. Thus when integrating $Z_k^\infty(z,\lambda)$ along contours starting and ending at the same points we get only contribution from the odd part of $Z_k^\infty(z,\lambda)$, i.e. from $\sigma_k Z_\infty^-(z,\lambda)$.

5. High energy limit of loci of zeros of fundamental solutions for the potential $(-\mathrm{i}\alpha z)^n$

It follows from the lemma that making λ_0 sufficiently large, i.e. $\lambda_0 \gg C_\epsilon$, we can make $\chi_k^\infty(z,\lambda)$ arbitrarily close to unity for $z \in D_{k,\epsilon}$ and $|\lambda| > \lambda_0$. But this means that for such conditions $\psi_k^\infty(z,\lambda)$ vanishes nowhere in $D_{k,\epsilon}$. Therefore, we have the following theorem about loci of zeros of $\psi_k^\infty(z,\lambda)$.

Theorem 1. *For sufficiently large λ all zeros of $\psi_k^\infty(z,\lambda)$ lie entirely in the completion $C_{\text{cut}}\backslash D_{k,\epsilon}$ of the domain $D_{k,\epsilon}$.*

In the case of a non-critical SG this completion coincides of course with $V_{k,\epsilon}$ while in the critical case it can contain also a whole sector. This is because in the case of the potential considered a connected set of SL's which contains more than three lines with three of them emerging from the same turning point z_k has to contain also one more turning point, i.e. z_{-k}. If we consider a solution $\psi_k^\infty(z,\lambda)$ defined in the sector S_k then the sector S_{-k} and its boundary ∂S_{-k} cannot be connected with S_k by canonical paths, i.e. they lie in the completion $C_{\text{cut}}\backslash D_k$. We see that S_{-k} is just the sector which has been mentioned earlier as contained in $C_{\text{cut}}\backslash D_{k,\epsilon}$ (see figure 3).

To formulate a theorem giving the precise positions of the roots of $\psi_k^\infty(z,\lambda)$ mentioned in theorem 1 for the cases of non-critical SG's let us make the following several notes.

The first one is that in choosing C_{cut} we can always choose the corresponding cuts in such a way to ensure all the SL's lying in C_{cut} to be strictly (i.e. not quasi) continuous in this cut plane, i.e. none of these lines can cross any cut. With such a choice for every root z_k of $W_n^\infty(z)$ the values of $\mathrm{Im}\big(\lambda\int_{z_k}^z\sqrt{W_n^\infty(y)}\,\mathrm{d}y\big)$ have definite sign on every SL emerging from z_k depending on the line.

The second one is, as we have already mentioned, that in the case of non-critical SG's every turning point z_l belongs to ∂D_k together with exactly one of its three SL's which emerge from it and ∂D_k is collected exactly of only such SL's.

The third one is the most important since it establishes the way of taking the limit $\lambda \to \infty$. Namely, as it follows from appendix B zeros of $\psi_k^\infty(z,\lambda)$ scatter around such an exceptional line sliding down in the limit considered in directions to the respective turning point from which the line emerges. If these zeros are not sufficiently far from the turning point they unavoidably approach their limit just in this point. However in such a case the conditions of the validity of the representations (13)–(16) and their semiclassical expansions (18)–(21) are broken, i.e. the limit positions of zeros are outside of the domains $D_{k,\epsilon}$ engaged in taking the limit. A necessary condition to get the limit loci of zeros different from the turning point is to take their initial loci to be λ-dependent. This dependence in the formulae given below is not only necessary but also sufficient giving finite loci of zeros different from turning points.

This dependence however defines also the way of getting the limit $\lambda \to \infty$ enforcing to put $|\lambda| = [|\lambda|] + \Lambda$, $0 \leqslant \Lambda < 1$, where $[|\lambda|]$ is a step function of $|\lambda|$ (i.e. an integer not greater than $|\lambda|$ itself) and to consider the limit $\lambda \to \infty$ conditioned by the fixed value of Λ. This kind of limit will be called *regular* to distinguish it from the *free* limit, i.e. with no conditions. Obviously, each Λ defines a different way of getting the limit $\lambda \to \infty$ by $\psi_k^\infty(z,\lambda)$ and its zeros as well.

Nevertheless, we can also consider zeros, the limit loci of which are just turning points when $\lambda \to \infty$. However we have to remember in such cases that the asymptotic formulae used to get these loci have to be bounded to the domains $D_{k,\epsilon}$.

The following theorems give the precise limit positions of roots of $\psi_k^\infty(z,|\lambda|\,\mathrm{e}^{\mathrm{i}\beta})$ for $|\lambda| \to \infty$ up to *all* orders of λ^{-1}.

Theorem 2a. *Zeros* $\zeta_{l,qr}^{(k)}(\lambda)$, $|\lambda| = [|\lambda|] + \Lambda$, $l \in A_{n,\alpha}$, $q = 0,1,2,\ldots$, $r = 0,\pm1,\pm2,\ldots$, *of* $\psi_k^\infty(z,|\lambda|\mathrm{e}^{\mathrm{i}\beta})$, $0 < |\beta| < \frac{\pi}{n}$, *i.e. in the non-critical cases, in the regular limit* $[|\lambda|] \to \infty$, *are distributed on* C_{cut} *uniquely along the corresponding exceptional SL's according to the formulae:*

$$\int_{K_l(\zeta_{l,qr}^{(k)}(\lambda))}\left(\frac{1}{2}\sqrt{W_n^\infty(y)} - \frac{1}{2\lambda}Z_k^\infty(y,\lambda)\right)\mathrm{d}y = \pm\left(q[|\lambda|] + r - \frac{1}{4}\right)\frac{\mathrm{i}\pi}{\lambda} \tag{26}$$

where $K_l(\zeta_{l,qr}^{(k)}(\lambda))$ *is a contour which starts and ends at* $\zeta_{l,qr}^{(k)}(\lambda)$ *rounding the turning point* z_l *anticlockwise and* $Z_k^\infty(z,\lambda)$ *is defined by (20).*

Zeros $\zeta_{l,qr}^{(k)}(\lambda)$ *have the following semiclassical expansion:*

$$\zeta_{l,qr}^{(k)}(\lambda) = \sum_{p\geqslant 0}\frac{1}{\lambda^p}\zeta_{l,qr;p}^{(k)}(\Lambda) \tag{27}$$

for $q > 0$ *with two first terms given by:*

$$\begin{aligned}
&\int_{K_l(\zeta_{l,qr0}^{(k)})}\frac{1}{2}\sqrt{W_n^\infty(y)}\,\mathrm{d}y = \int_{z_l}^{\zeta_{l,qr;0}^{(k)}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y = \pm q\mathrm{i}\pi\,\mathrm{e}^{-\mathrm{i}\beta}\\
&\zeta_{l,qr;1}^{(k)}(\Lambda) = \pm\left(r - q\Lambda - \frac{1}{4}\right)\frac{\mathrm{i}\pi\,\mathrm{e}^{-\mathrm{i}\beta}}{\sqrt{W_n^\infty\big(\zeta_{l,qr;0}^{(k)}\big)}}
\end{aligned} \tag{28}$$

For $q = 0$ *we have instead* $\zeta_{l,0r;0}^{(k)}(\Lambda) \equiv z_l$ *and*

$$\begin{aligned}
&\int_{z_l}^{z_l+\zeta_{l,0r;1}^{(k)}(\Lambda)/\lambda}\sqrt{W_n^\infty(y)}\,\mathrm{d}y = \left(r - \frac{1}{4}\right)\frac{\mathrm{i}\pi}{\lambda}\\
&r > \frac{|\lambda|}{\pi}\limsup_{|\phi|\leqslant\pi}\left|\int_{z_l}^{z_l+\epsilon\,\mathrm{e}^{\mathrm{i}\phi}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y\right|
\end{aligned} \tag{29}$$

as well as

$$\zeta^{(k)}_{l,0r;2}(\Lambda) = \frac{1}{8}\frac{\int_{K_l\left(z_l+\zeta^{(k)}_{l,0r;1}(\Lambda)/\lambda\right)} X_1^\infty(y)\,\mathrm{d}y}{\sqrt{W_n^\infty\left(z_l+\zeta^{(k)}_{l,0r;1}(\Lambda)/\lambda\right)}} \tag{30}$$

The signs $\pm$ *above are to be chosen the same as of* $\mathrm{Im}\left(\mathrm{e}^{\mathrm{i}\beta}\int_{z_l}^{z}\sqrt{W_n^\infty(y)}\,\mathrm{d}y\right)$ *on the corresponding exceptional lines.*

First of all what follows from the formulae (27) and (28) is that, according to (5), $\zeta^{(k)}_{l,qr0}$, $q = 1, 2, \ldots$, lie on SL emerging from z_l while the same cannot be said about the first term of the expansion (27), i.e. the integral $\int_{z_l}^{\zeta^{(k)}_{l,qr}(\lambda)}\sqrt{W_n^\infty(y)}\,\mathrm{d}y$ is not purely imaginary if the first and next orders of λ^{-1} are included in the expansion (27). It means that zeros $\zeta^{(k)}_{l,qr}(\lambda)$ deviate from SL considered when higher order terms in (27) are included beginning from the first one.

We get therefore the following picture of loci of zeros $\zeta^{(k)}_{l,qr}(\Lambda)$ in the considered regular limit.

For each Λ, $0 \leqslant \Lambda < 1$, these zeros are distributed along the SL emerging from z_l.

However for $q = 0$ as we have mentioned earlier only those zeros of $\psi_k^\infty(z,\lambda)$ can be considered which lie in the domain $D_{k,\epsilon}$, i.e those whose distances from the turning points $z_l, l \in A_{n,\alpha}$, cannot be smaller than ϵ. The semiclassical method we have used does not permit us to study zeros of $\psi_k^\infty(z,\lambda)$ which lie arbitrarily close to z_l. In fact as it follows from the discussion in appendix B all zeros of $\psi_k^\infty(z,\lambda)$ lying at finite distances from $z_l, l \in A_{n,\alpha}$, before the limit $\lambda \to \infty$ is taken aggregate around these turning points arbitrarily close to them when the limit is done. Therefore the semiclassical method allows us to observe only 'tails' of these groups of zeros and these tails are described by formulae (29)–(30). Since as we have seen the case $q = 0$ cannot be fully included into our discussion we shall ignore it in the remaining theorems except the next one.

For $q > 0$ zeros detected in these cases come from a vicinity of the infinite point of the C_{cut}-plane. For a given q, $q = 1, 2, \ldots$, every two neighbouring zeros are separated from each other by $\mathrm{i}\pi\,\mathrm{e}^{-\mathrm{i}\beta}\big/\left(\lambda\sqrt{W_n^\infty(\zeta^{(k)}_{l,qr0})}\right)$. For a given q, $q = 1, 2, \ldots$, and r, $r = 0, \pm1, \pm2, \ldots$, when Λ changes from zero to unity these zeros occupy a line

$$z = \zeta^{(k)}_{l,qr;0} + \left(r - q\Lambda - \frac{1}{4}\right)\frac{\mathrm{i}\pi\,\mathrm{e}^{-\mathrm{i}\beta}}{\sqrt{W_n^\infty\left(\zeta^{(k)}_{l,qr;0}\right)}}\frac{1}{[|\lambda|]+\Lambda} \tag{31}$$

which crosses the exceptional SL considered at $z = \zeta^{(k)}_{l,qr;0}$ for $\Lambda = \left(r - \frac{1}{4}\right)\big/q$.

On the other hand for higher $q = 1, 2, \ldots$, a distance d_q between the neighbouring such lines of zeros measured along the exceptional SL (it is L_k^l this time) is equal to $d_q = \mathrm{i}\pi\mathrm{e}^{-\mathrm{i}\beta}\big/\sqrt{W_n^\infty\left(\zeta^{(k)}_{l,q0;0}\right)}$ and vanishes with $q \to \infty$.

Figure 4(a) illustrates the exceptional lines for $\psi^\infty_{\frac{n+2}{2}}(z)$ for even n and $\alpha = \mathrm{e}^{-\mathrm{i}\frac{\pi}{n}}$ occupied by zeros of this solution with the distinguished exceptional line $L^{\frac{n}{4}-1}_{\frac{n+2}{2}}$ on which the described above details of the zeros distribution are shown.

To formulate the corresponding theorem for the critical cases of SG's, i.e. for real λ, we have to note that $C_{\mathrm{cut}}\backslash D_k = \partial D_k \cup S_{-k}$ so that ∂D_k contains the point z_{-k} together with the three SL's which emerge from it. This is just a set L_k^{-k} of exceptional SL's emerging from z_{-k}.

Let us note further a role played in the critical cases of SG's by the integral $\lambda\int_{z_k}^{z_{-k}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y \equiv \lambda I_k$. For λ real and positive it can always be represented as

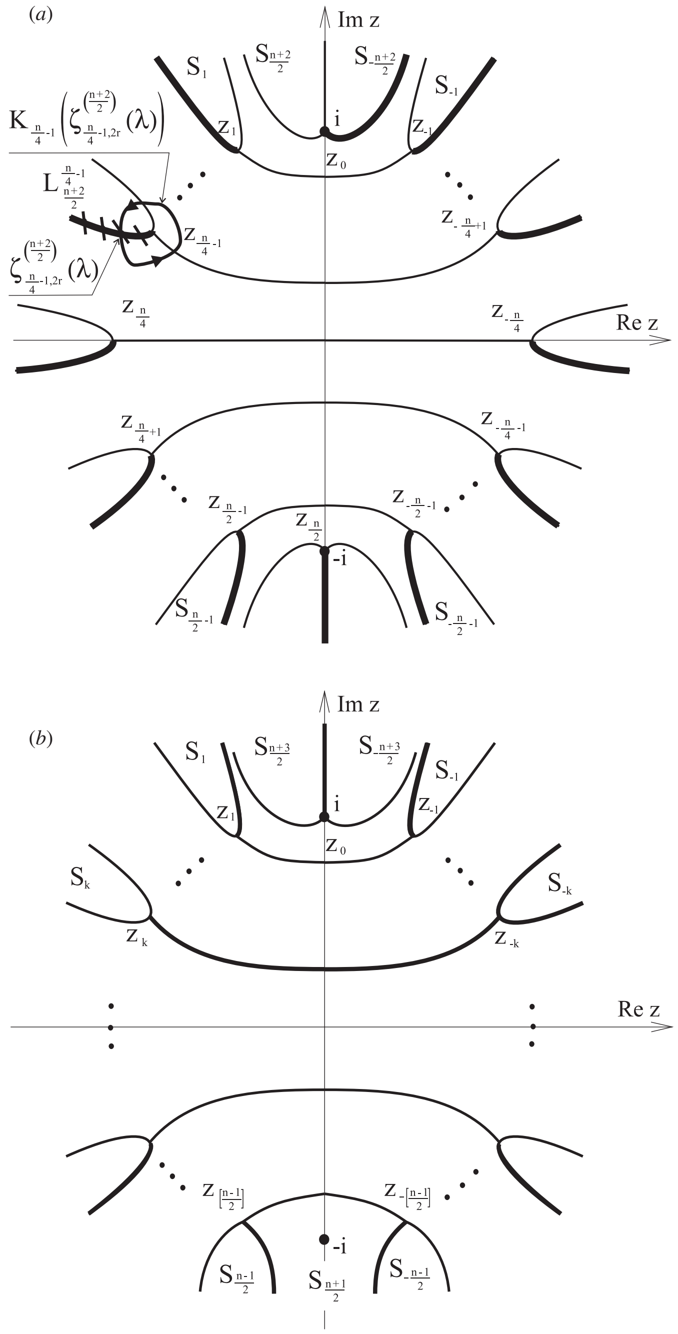

Figure 4. (*a*) SG for an even n and $\alpha = 1$. The bold lines are exceptional SL's along which zeros of $\psi^{\infty}_{\frac{n+2}{2}}(z, \lambda)$ are distributed in the regular limit $\lambda \to \infty$. On one of them details of such distribution are shown. (*b*) SG for an odd n and $\alpha = 1$. Bold lines are exceptional SL's along which zeros of $\psi^{\infty}_{\frac{n+2}{2}}(z, \lambda)$ are distributed in the regular limit $\lambda \to \infty$.

$I_k\lambda = -(r + R)\mathrm{i}\pi$ where $r \geqslant 0$ is an integer and $|R| \leqslant \frac{1}{2}$. Both r and R depend, of course, on λ and due to this dependence R can take on an arbitrary value from its domain of variation. Conversely, fixing R on some R_0 we choose in this way an infinite sequence $\lambda_{k,r}(R_0) = -\frac{\mathrm{i}\pi}{I_k}(r + R_0), r = 0, 1, 2, \ldots$ of λ's growing to infinity and such that $R(\lambda_{k,r}(R_0)) = R_0$. We shall maintain further for such sequences a description *regular* as well as for the limit $\lambda \to \infty$ itself using these sequences if $|R| < \frac{1}{2}$ while we call them *singular* if $|R| = \frac{1}{2}$. As previously all other sequences of λ's not restricted by any condition will be again called *free*.

It should be stressed however that as it follows from the form of the χ^∞-factors of FS's and what will be seen later when semiclassical limit $\lambda \to \infty$ of particular formulae will be taken that to apply regular limits one needs to satisfy the inequality $\cos(R\pi) \gg \frac{\lambda_0}{\lambda} > 0$. However this condition is always satisfied for any $|R| < \frac{1}{2}$ when λ is sufficiently large.

In theorem 2a we have used regular sequences of λ conditioned by fixed value of Λ. In theorem 2b below the regular limit $\lambda \to \infty$ will be also considered when R has a fixed value. This is essentially the main difference between the critical and non-critical cases in taking the limit $\lambda \to \infty$.

Theorem 2b. *Zeros $\zeta_{l,qr}^{(k)}(\lambda), |\lambda| = [|\lambda|] + \Lambda, l \in A_{n,\alpha}, l \neq -k, q = 1, 2, \ldots, r = 0, \pm1, \pm2, \ldots,$ of $\psi_k^\infty(z, |\lambda|)$, i.e. in the critical cases, in the regular limit $[|\lambda|] \to \infty$, i.e. with fixed Λ, are distributed on C_{cut} uniquely along the corresponding exceptional SL's according to the formulae:*

$$\int_{K_l(\zeta_{l,qr}^{(k)}(\lambda))} \left(\frac{1}{2}\sqrt{W_n^\infty(y)} - \frac{1}{2\lambda} Z_k^\infty(y, \lambda)\right) \mathrm{d}y = \pm\left(q[|\lambda|] + r - \frac{1}{4}\right)\frac{\mathrm{i}\pi}{\lambda} \tag{32}$$

where $K_l\big(\zeta_{l,qr}^{(k)}(\lambda)\big)$ is a contour which starts and ends at $\zeta_{l,qr}^{(k)}(\lambda)$ rounding the turning point z_l anticlockwise.

Zeros $\zeta_{l,qr}^{(k)}(\lambda)$ have the following semiclassical expansion:

$$\zeta_{l,qr}^{(k)}(\lambda) = \sum_{p\geqslant 0} \frac{1}{\lambda^p} \zeta_{l,qr;p}^{(k)}(\Lambda) \tag{33}$$

for $q > 0$ with two first terms given by

$$\begin{aligned} &\int_{K_l\left(\zeta_{l,qr;0}^{(k)}\right)} \frac{1}{2}\sqrt{W_n^\infty(y)}\,\mathrm{d}y = \int_{z_l}^{\zeta_{l,qr;0}^{(k)}} \sqrt{W_n^\infty(y)}\,\mathrm{d}y = \pm q\mathrm{i}\pi \\ &\zeta_{l,qr;1}^{(k)}(\Lambda) = \pm\left(r - q\Lambda - \frac{1}{4}\right)\frac{\mathrm{i}\pi}{\sqrt{W_n^\infty\big(\zeta_{l,qr;0}^{(k)}\big)}}. \end{aligned} \tag{34}$$

For $q = 0$ we have instead $\zeta_{l,0r;0}^{(k)}(\Lambda) \equiv z_l$ and

$$\begin{aligned} &\int_{z_l}^{z_l+\zeta_{l,0r;1}^{(k)}(\Lambda)/\lambda} \sqrt{W_n^\infty(y)}\,\mathrm{d}y = \left(r - \frac{1}{4}\right)\frac{\mathrm{i}\pi}{\lambda} \\ &r > \frac{|\lambda|}{\pi} \limsup_{|\phi|\leqslant\pi} \left|\int_{z_l}^{z_l+\epsilon\mathrm{e}^{\mathrm{i}\phi}} \sqrt{W_n^\infty(y)}\,\mathrm{d}y\right| \end{aligned} \tag{35}$$

as well as

$$\zeta_{l,0r;2}^{(k)}(\Lambda) = \frac{1}{8}\frac{\int_{K_l(z_l+\zeta_{l,0r;1}^{(k)}(\Lambda)/\lambda)} X_1^\infty(y)\,\mathrm{d}y}{\sqrt{W_n^\infty\big(z_l + \zeta_{l,0r;1}^{(k)}(\Lambda)/\lambda\big)}}. \tag{36}$$

The signs $\pm$ above are to be chosen the same as of $\mathrm{Im}\left(\int_{z_l}^{z}\sqrt{W_n^\infty(y)}\,dy\right)$ *on the corresponding exceptional lines.*

In the case $l = k$ the number q is bounded, i.e. $q \leqslant |I_k|/\pi$ and in (34) the minus sign is to be chosen according to our earlier conventions.

Additionally in a regular limit $\lambda_{k,s} \to \infty$, i.e. with R fixed, where $\lambda_{k,s} = -\frac{s+R}{I_k}i\pi = [\lambda_{k,s}] + \Lambda_{k,s}(R)$, $s = 0, 1, 2, \ldots$, there are two infinite sequences of zeros $\zeta^{(k)\pm}_{-k,qr}$, $q = 1, 2, \ldots, r = 0, \pm 1, \pm 2, \ldots$, of $\psi_k^\infty(z, |\lambda|)$ distributed along the two infinite SL's of the sector S_{-k} according to the following rules:

$$\int_{z_{-k}}^{\zeta^{(k)\pm}_{-k,qr}} \sqrt{W_n^\infty(y)}\,dy = -\left(q[\lambda_{k,s}] + r - \frac{1}{4} + \frac{R}{2}\right)\frac{i\pi}{\lambda_{k,s}} + \frac{1}{4\lambda_{k,s}}\oint_{K_k} Z_k^\infty\,dy$$
$$\pm\frac{1}{2\lambda_{k,s}}\ln 2\cos\left(R\pi + \frac{1}{2}\mathrm{Im}\oint_{K_k} Z_k^\infty\,dy\right) \pm \frac{1}{2\lambda_{k,s}}\int_{K_{-k}(\zeta^{(k)\pm}_{-k,qr})} Z_k^\infty\,dy \tag{37}$$

with the following first coefficients of the corresponding semiclassical expansion of $\zeta^{(k)\pm}_{-k,q;r}$:

$$\begin{aligned}&\int_{z_{-k}}^{\zeta^{(k)\pm}_{-k,qr;0}} \sqrt{W_n^\infty(y)}\,dy = -qi\pi\\ &\zeta^{(k)\pm}_{-k,qr;1}(R) = -\left(r - q\Lambda_{k,s}(R) - \frac{1}{4} + \frac{R}{2} \mp \frac{1}{2}\ln 2\cos(R\pi)\right)\frac{i\pi}{\sqrt{W_n^\infty\left(\zeta^{(k)\pm}_{-k,qr;0}\right)}}\end{aligned} \tag{38}$$

where the plus sign corresponds to a vicinity of the SL being the upper boundary of S_{-k} while the minus one to a vicinity of its lower boundary.

Figure 4(*b*) shows all the exceptional SL's of the odd n case occupied by zeros of $\psi_k^\infty(z, \lambda)$ in the limits considered in the above theorem.

All the above three theorems have been proved in appendix B.

It follows therefore from the notes made above that the cases $R = \pm\frac{1}{2}$ have to be considered separately. It should not be surprising that they are just the quantized cases of the solution $\psi_k^\infty(z, \lambda)$ considered.

6. High energy limit of loci of zeros of fundamental solutions for the potential $W_n(z, \lambda) + 1$

Let us denote $-i\alpha z$ by ζ and $\lambda^{-\frac{2}{n+2}}$ by η and let $|\arg\lambda| < \frac{\pi}{n}$. Then for the polynomial $W_n(z, \lambda)$ we have

$$\begin{aligned}&W_n(z, \lambda) \to \tilde{W}_n(\zeta, \eta) \equiv W_n\left(i\alpha^{-1}\zeta, \eta^{-\frac{n+2}{2}}\right) = \zeta^n - 1 + \sum_{k=1}^{n-1} b'_{n-k}\eta^k\zeta^{n-k}\\ &b'_{n-k} = b_{n-k}(-i\alpha)^{\frac{2k}{n+2}}.\end{aligned} \tag{39}$$

Let us note that bounding $|\arg\lambda|$ by $\frac{\pi}{n}$ we always get the standard SG's corresponding to $W_n^\infty(z)$ as a limit of the corresponding SG's related to $W_n(z, \lambda)$ when $\lambda \to \infty$.

It follows from the above form of $\tilde{W}_n(\zeta, \eta)$ that if ζ_0 is its root for $\eta = 0$ then, by the implicit function theorem, there is also such a root $\zeta(\eta)$ of $\tilde{W}_n(\zeta, \eta)$ which is close to ζ_0 for η sufficiently close to zero, i.e. $\zeta(\eta) = \zeta_0 - \frac{1}{n}b'_{n-1}\eta + O(\eta^2)$ for $\eta \to 0$. It means of course that for sufficiently large λ all the roots of $W_n(z, \lambda)$ are simple and are close to the corresponding roots of $W_n^\infty(z)$, i.e to each root z_k of the latter potential there is the root $z_k(\lambda)$ of the former one such that $z_k(\lambda) = z_k - i\frac{1}{n\alpha}b'_{n-1}\lambda^{-\frac{2}{n+2}} + O\left(\lambda^{-\frac{4}{n+2}}\right)$ for $|\lambda| \to \infty$, $k \in A_{n,\alpha}$.

Let us re-examine for the potential considering the respective notions of SG's and FS's and remaining ones related to them. A necessity to do it is caused by the λ-dependence of the potential $W_n(z,\lambda)+1$ and as it is seen from (39) this dependence is by powers of $\lambda^{-\frac{2}{n+2}}$. Such a dependence complicates an analysis of λ-dependence of SG's, FS's and other notions. It is easier to make the respective analyses by a substitution $\lambda \to \lambda^{\frac{n+2}{2}}$ in the corresponding Schrödinger equation (2) with the potential $W_n(z,\lambda)+1$ by which the potential itself convert to depend on natural powers of λ^{-1} and making simultaneously the same substitution for the SE with the potential $W_n^\infty(z)+1$.

The corresponding SG's can be now defined by using the new λ-parameter for both the potentials. It does not disturb much properties of SG's for the potential $W_n^\infty(z)+1$ except that it makes these graphs more sensitive for λ-changes (because λ is now powered by $(n+2)/2$), i.e. the graphs come back to their initial forms when λ is rotated by the angle $\phi = 2\pi/(n+2)$ rather than by π as previously, for example, and this is just this new angle which is now a period for such graphs' changes.

In the case of the potential $W_n(z,\lambda)+1$ the corresponding graphs defined by (5) where λ is substituted by $\lambda^{\frac{n+2}{2}}$ depend now on λ and if λ changes it needs a total change equal to 2π of its argument to make the graphs coming back to their initial positions. Changing absolute value of λ alone also forces the graphs to change while the graphs corresponding to $W_n^\infty(z)+1$ are insensitive on such a change of λ.

All these do not prevent us however to define the corresponding FS's by formulae (13)–(16) together with the corresponding definitions of canonical paths, canonical domains, etc, including also constructions of domains $D_{k,\epsilon}$ contained in canonical domains D_k together with ϵ-vicinities $V_{k,\epsilon}$ of their respective boundaries ∂D_k, $k \in B_{n,\alpha}$, and with *exceptional* SL's related to them. We have only to remember that all the above notions depend now on λ and can change with it continuously.

Nevertheless, despite the mentioned λ-dependence lemma of section 4 remains valid also in the case of the potential $W_n(z,\lambda)+1$. This can be argued as follows.

Consider a fundamental solution $\psi_k(z,\lambda)$ to the SE (2) corresponding to the potential $W_n(z,\lambda)+1$ and a respective domain $D_{k,\epsilon}(\lambda)$ where this solution can be continued canonically to any of its point and let $|\lambda| > \lambda_0$ with λ_0 sufficiently large to find each $z_k(\lambda)$ inside corresponding circle vicinities $\Delta_k(\epsilon)$, $k \in B_{n,\alpha}$.

By the same arguments and with obvious changes the inequality (17) in the domain $D_{k,\epsilon}(\lambda)$ can be written for the solution $\psi_k(z,\lambda)$ as follows:

$$\begin{aligned} &|\chi_k^\infty(z,\lambda)-1| \leqslant \exp\left[\frac{C_\epsilon(\lambda)}{\lambda_0^{\frac{n+2}{2}}}\right]-1 \\ &C_\epsilon(\lambda) = \liminf_{\gamma_k(z),z\in C_{\mathrm{cut}}(\epsilon),k\in B_{n,\alpha}} \int_{\gamma_k(z)} |\omega(\xi,\lambda)\,\mathrm{d}\xi| < \infty. \end{aligned} \tag{40}$$

Therefore lemma of section 4 sounds now (see appendix B for a proof):

Lemma'. *In the domain $D_{k,\epsilon}(\lambda)$ the factor $\chi_k^\infty(z,\lambda)$ of the FS solution $\psi_k(z,\lambda)$ constructed for the potential $W_n(z,\lambda)+1$ satisfies the following bound:*

$$\begin{aligned} &|\chi_k^\infty(z,\lambda)-1| \leqslant \exp\left[\frac{C_\epsilon}{\lambda_0^{\frac{n+2}{2}}(\epsilon)}\right]-1, \qquad |\lambda| > \lambda_0(\epsilon) \\ &C_\epsilon = C_\epsilon^\infty + \epsilon \end{aligned} \tag{41}$$

with $\lambda_0(\epsilon)$ sufficiently large and with fixed but arbitrary small ϵ.

Therefore theorem 1 remains valid also for the potential $W_n(z,\lambda)+1$ with no changes.

To formulate analogues of theorem 2a and 2b we have to establish the corresponding forms of the semiclassical expansions (18)–(20) for the case of the potential $W_n(z,\lambda)+1$. These expansions have to be a little bit different because of a dependence of the potential on λ.

First we note that the χ-factors of the FS's $\psi_k(z,\lambda)$ with the potential $W_n(z,\lambda)+1$ and λ changed as mentioned earlier satisfy the following equation:

$$W_n^{-\frac{1}{4}}(z,\lambda)\left(W_n^{-\frac{1}{4}}(z,\lambda)\chi_k(z,\lambda)\right)''+2\sigma_k\lambda^{\frac{n+2}{2}}\chi_k'(z,\lambda)=0 \tag{42}$$

which can be transformed to the (pseudo-)integral equation of the form

$$\chi_k(z,\lambda)=1-\frac{\sigma_k}{2\lambda^{\frac{n+2}{2}}}\int_{\infty_k}^{z}W_n^{-\frac{1}{4}}(y,\lambda)\left(W_n^{-\frac{1}{4}}(y,\lambda)\chi_k(z,\lambda)\right)''\,\mathrm{d}y \tag{43}$$

corresponding to the χ_k-factor of the FS defined in the sector $S_k(\lambda)$.

An attempt to get a solution for $\chi_k(z,\lambda)$ from (43) by iterations leads to a divergent series which however appears to be just a semiclassical expansion for $\chi_k(z,\lambda)$ of the form:

$$\chi_k(z,\lambda)\sim\chi_k^{as}(z,\lambda)=\sum_{p\geqslant 0}\left(-\frac{\sigma_k}{2\lambda^{\frac{n+2}{2}}}\right)^p\tilde{Y}_{k,p}(z,\lambda), \tag{44}$$

where $\tilde{Y}_{k,p}(z,\lambda)$ are given by (19) with $W_n^{\infty}(z)$ substituted by $W_n(z,\lambda)$.

If further using (19) (with the substitutions mentioned) we make the following expansion:

$$\begin{aligned}&\tilde{Y}_{k,p}(z,\lambda)=\sum_{q\geqslant 0}\tilde{Y}_{k;p,q}(z)\lambda^{-q}\\&\tilde{Y}_{k;0,q}(z)\equiv\delta_{0q}\end{aligned} \tag{45}$$

then (44) can be given the following forms depending on a parity of n:

$$\chi_k^{as}(z,\lambda)=\begin{cases}\sum_{p\geqslant 0}\sum_{q=0}^{m}\dfrac{Y_{k;pq}(z)}{\lambda^{(m+1)p+q}}, & n=2m\\[2ex] \sum_{p\geqslant 0}\dfrac{1}{2^{2p}}\displaystyle\sum_{q=0}^{2m+2}\dfrac{Y_{k;2p,q}(z)}{\lambda^{(2m+3)p+q}} & \\[2ex] -\dfrac{\sigma_k}{2\lambda^{\frac{2m+3}{2}}}\displaystyle\sum_{p\geqslant 0}\dfrac{1}{2^{2p}}\sum_{q=0}^{2m+2}\dfrac{Y_{k;2p+1,q}(z)}{\lambda^{(2m+3)p+q}}, & n=2m+1\end{cases} \tag{46}$$

where the coefficients of the last expansions are given by those of the expansion (45) as

$$\begin{aligned}&Y_{k;pq}(z)=\sum_{r=0}^{p}\left(-\frac{\sigma_k}{2}\right)^r\tilde{Y}_{k;r,(m+1)(p-r)+q}(z), && n=2m\\&\left.\begin{aligned}&Y_{k;2p,q}(z)=\sum_{r=0}^{p}2^{2(p-r)}\tilde{Y}_{k;2r,(2m+3)(p-r)+q}(z)\\&Y_{k;2p+1,q}(z)=\sum_{r=0}^{p}2^{2(p-r)}\tilde{Y}_{k;2r+1,(2m+3)(p-r)+q}(z)\end{aligned}\right\} && n=2m+1.\end{aligned} \tag{47}$$

The exponential representations of $\chi_k^{as}(z,\lambda)$ have similar forms to (18), i.e.

$$\chi_k^{as}(z,\lambda) = \exp\left(\int_{\infty_k}^{z} Z_k(y,\lambda)\,\mathrm{d}y\right)$$

$$Z_k(z,\lambda) = \begin{Bmatrix} \displaystyle\sum_{p\geqslant 1}\sum_{q=0}^{m}\frac{X_{k;pq}(z)}{\lambda^{(m+1)p+q}}, & n=2m \\ \displaystyle\sum_{p\geqslant 1}\frac{1}{2^{2p}}\sum_{q=0}^{2m+2}\frac{X_{k;2p,q}(z)}{\lambda^{(2m+3)p+q}} & \\ \displaystyle -\frac{\sigma_k}{2\lambda^{\frac{2m+3}{2}}}\sum_{p\geqslant 0}\frac{1}{2^{2p}}\sum_{q=0}^{2m+2}\frac{X_{k;2p+1,q}(z)}{\lambda^{(2m+3)p+q}}, & n=2m+1 \end{Bmatrix}$$

$$= \frac{1}{\chi_k^{as}(z,\lambda)}\frac{\mathrm{d}\chi_k^{as}(z,\lambda)}{\mathrm{d}z}. \tag{48}$$

However the corresponding recurrent relations for $X_{k;pq}(z)$ are now much more complicated. Nevertheless, since $Z_k(z,\lambda)$ satisfy the equations

$$W_n^{-\frac14}\big(W_n^{-\frac14}\big)'' + 2\big(W_n^{-\frac14}\big(W_n^{-\frac14}\big)' + \sigma_k\lambda^{\frac{n+2}{2}}\big)Z_k + W_n^{-\frac12}\big(Z_k^2 + Z_k'\big) = 0 \tag{49}$$

then the partition $Z_k(z,\lambda) = Z^+(z,\lambda) + \sigma_k Z^-(z,\lambda)$ still can be done with the same properties (24) for $Z^+(z,\lambda)$ and the corresponding conclusions about its analytical properties on the z-plane.

Unlike the case of the potential $W_n^\infty(z)+1$ we have to use also the following asymptotic expansion of $\sqrt{W_n(z,\lambda)}$

$$\sqrt{W_n(z,\lambda)} = \sqrt{W_n^\infty(z)} + \sum_{p\geqslant 1}\frac{W_{n,p}(z)}{\lambda^p} \tag{50}$$

as well as the asymptotic expansions of the limit loci of zeros $\zeta_l^{(k)}(\lambda), l\in A_{n,\alpha}$, of the FS $\psi_k(z,\lambda)$

$$\zeta_l^{(k)}(\lambda) = \begin{cases} \displaystyle\sum_{p\geqslant 0}\sum_{q=0}^{m}\frac{\zeta_{l;p,q}^{(k)}}{\lambda^{(m+1)p+q}} & n=2m \\ \displaystyle\sum_{p\geqslant 0}\sum_{q=0}^{2m+2}\frac{\zeta_{l;2p,q}^{(k)}}{\lambda^{(2m+3)p+q}} & \\ \displaystyle +\frac{1}{\lambda^{\frac{2m+3}{2}}}\sum_{p\geqslant 0}\sum_{q=0}^{2m+2}\frac{\zeta_{l;2p+1,q}^{(k)}}{\lambda^{(2m+3)p+q}}, & n=2m+1. \end{cases} \tag{51}$$

Having done properly the semiclassical expansions of the respective quantities in order to be as close as possible to the previous formulations of theorem 2a and 2b, we can come back to the previous form of λ-dependence by a substitution in all the above formulae $\lambda^{\frac{n+2}{2}}$ back by λ itself. Then we can formulate the following theorems analogous to theorem 2a and 2b.

Theorem 3a. *In the non-critical case and in the regular limit* $\lambda\to\infty$ *zeros* $\zeta_{l,qr}^{(k)}(\lambda)$, $|\lambda| = [|\lambda|] + \Lambda, l\in A_{n,\alpha}, q = 1,2,\ldots, r = 0,\pm1,\pm2,\ldots,$ *of* $\psi_k(z,|\lambda|e^{i\beta}), 0<|\beta|<\frac{\pi}{n}$ *are distributed on* $C_{\rm cut}$ *uniquely along the corresponding exceptional SL's according to the formulae*

$$\int_{K_l(\zeta_{l,qr}^{(k)}(\lambda))}\left(\frac12\sqrt{W_n(y,\lambda)} - \frac{1}{2\lambda}Z_k(y,\lambda)\right)\mathrm{d}y = \pm\left(q[|\lambda|] + r - \frac14\right)\frac{\mathrm{i}\pi}{\lambda} \tag{52}$$

where $K_l\big(\zeta_{l,qr}^{(k)}(\lambda)\big)$ is a contour which starts and ends at $\zeta_{l,qr}^{(k)}(\lambda)$ rounding the turning point z_l anticlockwise and $Z_k(z,\lambda)$ is defined by (48).

The two lowest terms of the semiclassical expansions of zeros $\zeta_{l,qr}^{(k)}(\lambda)$ given by (49) are the following:

$$\begin{gathered}\int_{K_l(\zeta_{l,qr;0,0}^{(k)})} \frac{1}{2}\sqrt{W_n^\infty(y)}\,\mathrm{d}y = \int_{z_l}^{\zeta_{l,qr;0,0}^{(k)}} \sqrt{W_n^\infty(y)}\,\mathrm{d}y = \pm q\mathrm{i}\pi\,\mathrm{e}^{-\mathrm{i}\beta}\\ \zeta_{l,qr;0,1}^{(k)}(\Lambda) = \pm\left(r - q\Lambda - \frac{1}{4}\right)\frac{\mathrm{i}\pi\,\mathrm{e}^{-\mathrm{i}\beta}}{\sqrt{W_n^\infty\big(\zeta_{l,qr;0,0}^{(k)}\big)}}\end{gathered} \tag{53}$$

and are calculated from (52) according to formula (B.10) of appendix B.

The signs $\pm$ above are to be chosen to agree with signs of $\mathrm{Im}\big(\mathrm{e}^{\mathrm{i}\beta}\int_{z_l}^{z}\sqrt{W_n(y,\lambda)}\,\mathrm{d}y\big)$ *on the corresponding exceptional lines.*

Suppose now that there is an inner SL linking the roots z_{k_0}, z_{-k_0} of $W_n(z,\lambda)$ while the others are absent (see figure 5(*b*)). In the limit case $\lambda\to\infty$ it is possible only for $\arg\lambda\neq 0$ but with $\arg\lambda\sim|\lambda|^{-\frac{2}{n+2}}$ (see appendix C). Then for the assumed arrangement of SL's we have the following:

Theorem 3b. *In the critical case, when there is inner SL linking z_{k_0} with z_{-k_0}, and in the regular limit $\lambda\to\infty$ zeros $\zeta_{l,qr}^{(k)}(\lambda)$, $|\lambda| = [|\lambda|]+\Lambda$, $l\in A_{n,\alpha}$, $q=1,2,\ldots$, $r=0,\pm1,\pm2,\ldots$, of $\psi_k(z,|\lambda|\mathrm{e}^{\mathrm{i}\beta})$, $0<|\beta|<\frac{\pi}{n}$, $k\neq k_0,-k_0$, are distributed on C_{cut} uniquely along the corresponding exceptional SL's according to the formulae:*

$$\int_{K_l(\zeta_{l,qr}^{(k)}(\lambda))}\left(\frac{1}{2}\sqrt{W_n(y,\lambda)} - \frac{1}{2\lambda}Z_k(y,\lambda)\right)\mathrm{d}y = \pm\left(q[|\lambda|]+r-\frac{1}{4}\right)\frac{\mathrm{i}\pi}{\lambda} \tag{54}$$

where $K_l\big(\zeta_{l,qr}^{(k)}(\lambda)\big)$ is a contour which starts and ends at $\zeta_{l,qr}^{(k)}(\lambda)$ rounding the turning point z_l anticlockwise and with $Z_k(z,\lambda)$ given by (48)

The two lowest terms of the semiclassical expansions of zeros $\zeta_{l,qr}^{(k)}(\lambda)$ given by (49) are the following:

$$\begin{gathered}\int_{K_l(\zeta_{l,qr;0,0}^{(k)})} \frac{1}{2}\sqrt{W_n^\infty(y)}\,\mathrm{d}y = \int_{z_l}^{\zeta_{l,qr;0,0}^{(k)}} \sqrt{W_n^\infty(y)}\,\mathrm{d}y = \pm q\mathrm{i}\pi\,\mathrm{e}^{-\mathrm{i}\beta}\\ \zeta_{l,qr;0,1}^{(k)}(\Lambda) = \pm\left(r - q\Lambda - \frac{1}{4}\right)\frac{\mathrm{i}\pi\mathrm{e}^{-\mathrm{i}\beta}}{\sqrt{W_n^\infty\big(\zeta_{l,qr;0,0}^{(k)}\big)}}\end{gathered} \tag{55}$$

and are calculated from (52) according to formula (B.10) of appendix B.

The signs $\pm$ above are to be chosen to agree with signs of $\mathrm{Im}\big(\mathrm{e}^{\mathrm{i}\beta}\int_{z_l}^{z}\sqrt{W_n(y,\lambda)}\,\mathrm{d}y\big)$ *on the corresponding exceptional lines.*

In the cases $l=k_0,-k_0$ the numbers q are bounded, i.e. $q\leqslant|I_{k_0}|/\pi$ with the minus sign in (53) chosen by assumption.

Additionally in a regular limit $\lambda_{k_0,s}\to\infty$, i.e. with R fixed, where $\lambda_{k_0,s} = -\frac{s+R}{I_{k_0}}\mathrm{i}\pi = [\lambda_{k_0,s}]+\Lambda_{k_0,s}(R)$, $s=0,1,2,\ldots$, there are two infinite sequences of zeros $\zeta_{-k_0,qr}^{(k_0)\pm}$, $q=1,2,\ldots$, $r=0,\pm1,\pm2,\ldots$ of $\psi_{k_0}(z,\lambda)$ distributed along the two SL's of the sector S_{-k_0} according to the following rules:

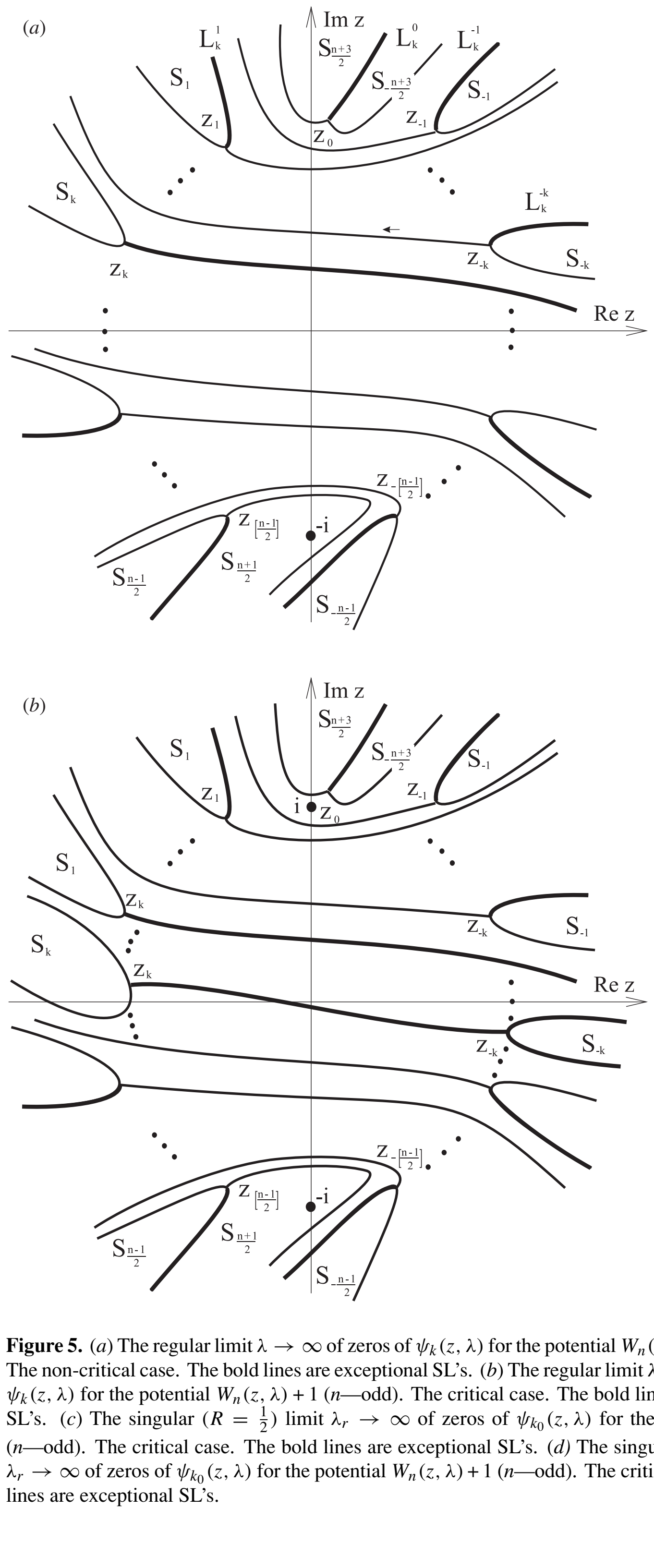

Figure 5. (*a*) The regular limit $\lambda \to \infty$ of zeros of $\psi_k(z, \lambda)$ for the potential $W_n(z, \lambda)+1$ (n—odd). The non-critical case. The bold lines are exceptional SL's. (*b*) The regular limit $\lambda \to \infty$ of zeros of $\psi_k(z, \lambda)$ for the potential $W_n(z, \lambda) + 1$ (n—odd). The critical case. The bold lines are exceptional SL's. (*c*) The singular ($R = \frac{1}{2}$) limit $\lambda_r \to \infty$ of zeros of $\psi_{k_0}(z, \lambda)$ for the potential $W_n^{\infty}(z)$ (n—odd). The critical case. The bold lines are exceptional SL's. (*d*) The singular ($R = \frac{1}{2}$) limit $\lambda_r \to \infty$ of zeros of $\psi_{k_0}(z, \lambda)$ for the potential $W_n(z, \lambda) + 1$ (n—odd). The critical case. The bold lines are exceptional SL's.

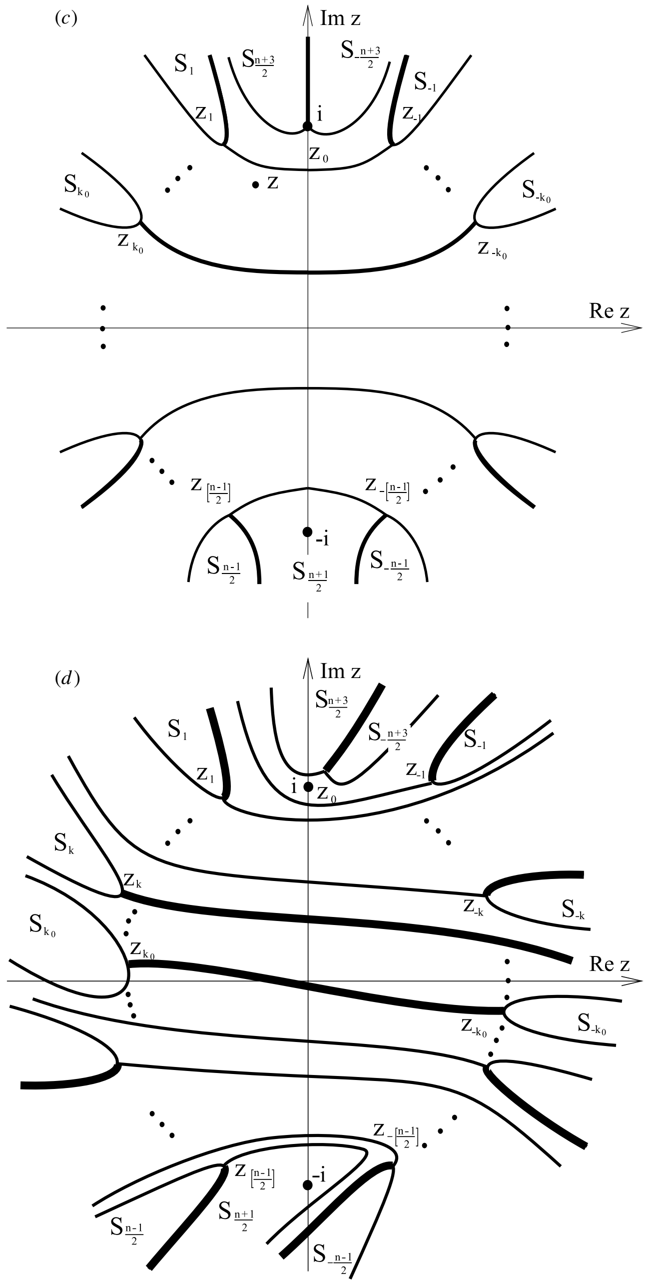

Figure 5. (Continued.)

$$\int_{z_{-k_0}}^{\zeta_{-k_0,qr}^{(k_0)\pm}} \sqrt{W_n(y,\lambda)}\,\mathrm{d}y = -\left(q[\lambda_{k_0,s}] + r - \frac{1}{4} + \frac{R}{2}\right)\frac{\mathrm{i}\pi}{\lambda_{k_0,s}} + \frac{1}{4\lambda_{k_0,s}}\oint_{K_{k_0}} Z_{k_0}\,\mathrm{d}y$$
$$\pm \frac{1}{2\lambda_{k_0,s}}\ln 2\cos\left(R\pi + \frac{1}{2}\mathrm{Im}\oint_{K_{k_0}} Z_{k_0}\,\mathrm{d}y\right) \pm \frac{1}{2\lambda_{k_0,s}}\int_{K_{-k_0}} \left(\zeta_{-k_0,qr}^{(k_0)\pm}\right) Z_{k_0}\,\mathrm{d}y \tag{56}$$

with the following lowest coefficients of the corresponding semiclassical expansion of $\zeta^{(k_0)\pm}_{-k_0,qr}$*:*

$$\int_{z_{-k_0}}^{\zeta^{(k_0)\pm}_{-k_0,qr;0,0}} \sqrt{W_n^\infty(y)}\,\mathrm{d}y = -q\mathrm{i}\pi$$

$$\zeta^{(k_0)\pm}_{-k_0,qr;0,1}(R) = -\left(r - q\Lambda_{k_0,s}(R) - \frac{1}{4} + \frac{R}{2} \mp \frac{1}{2}\ln 2\cos(R\pi)\right)\frac{\mathrm{i}\pi}{\sqrt{W_n^\infty\left(\zeta^{(k_0)\pm}_{-k_0,qr;0,0}\right)}} \tag{57}$$

where the plus sign corresponds to the SL being the upper boundary of S_{-k_0} *while the minus one to its lower boundary.*

Obviously the second part of this theorem applies also to the FS $\psi_{-k_0}(z,\lambda)$ *with appropriate changes.*

7. High energy behaviour of zeros for quantized energy

By a quantization of energy we mean an identification of some two chosen fundamental solutions. However such a choice is in fact quite limited particularly when the high energy limit, $|\lambda| \to \infty$, is taken.

First, in general, one cannot identify two neighboured FS's since they are always linearly independent.

Second, in the high energy limit, identifying two not neighboured FS's, one in fact demands, that the two sectors in which these FS's are defined have to be connected by an inner SL (see (59) below). This means that we can consider only the standard SG cases, i.e. when these inner SL's are parallel to the real axis of the C_{cut}-plane. Therefore we can put the energy quantization conditions only on the pairs $\psi_k^\infty(z,\lambda)$, $\psi_{-k}^\infty(z,\lambda)$, $k = \pm1, \pm2, \ldots, k \in A_{n,\alpha}$, for the potential $W_n^\infty(z)$ and for analogous pairs for the potential $W_n(z,\lambda)+1$. But as it follows from each of formulae (B.25) and (B.26) of appendix B the corresponding quantization conditions for the potential $W_n^\infty(z)$ when matching the pair $\psi_k^\infty(z,\lambda)$, $\psi_{-k}^\infty(z,\lambda)$ read

$$1 + \exp\left[2\int_{z_k}^{z_{-k}} \left(\sigma_k\lambda\sqrt{W_n^\infty(y)} + Z_k^\infty(y,\lambda)\right)\mathrm{d}y\right] = 0 \tag{58}$$

or

$$\int_{z_k}^{z_{-k}} \left(\sigma_k\lambda_s\sqrt{W_n^\infty(y)} + Z_k^\infty(y,\lambda_s)\right)\mathrm{d}y = -\left(s+\frac{1}{2}\right)\pi\mathrm{i}, \qquad s = 0,1,2,\ldots \tag{59}$$

what proves that in the limit $\lambda_s \to \infty$ the SL emerging from z_k ends at z_{-k}, i.e. this is the inner SL.

The equation (59) thus proves that for the potential $W_n^\infty(z)$ when the energy is quantized there are inner SL's for each $k = \pm1, \pm2, \ldots, k \in A_{n,\alpha}$, but only for one value of k, i.e. this which satisfies (59), the energy is quantized and the corresponding FS's coincide, i.e. $\psi_k^\infty(z,\lambda_s) = \mathrm{i}(-1)^s\psi_{-k}^\infty(z,\lambda_s)$ as it follows from (B.25) or (B.26). However in the case of the potential $W_n(z,\lambda)+1$ the SL satisfying equation (59) can be the unique inner one for the corresponding SG while the remaining SL's of this graph emerge to infinity.

Let now equation (59) be satisfied for $k = k_0$. Then comparing (59) with equation (37) we see that $R = \frac{1}{2}$ for this k_0 and therefore we cannot apply the result given by (37) to this case. Nevertheless, we can use the fact that now $\psi_{-k_0}^\infty(z,\lambda)$ coincides with $\psi_{k_0}^\infty(z,\lambda)$ up to a constant and since it cannot vanish in its sector S_{-k_0} so does $\psi_{k_0}^\infty(z,\lambda)$, i.e. there are no longer roots of $\psi_{k_0}^\infty(z,\lambda)$ along the boundaries of the sector S_{-k_0}.

Nevertheless, there are still such roots of $\psi_k^\infty(z,\lambda)$ in the corresponding sectors S_{-k} for $k \neq k_0$.

Let us call the fundamental solution $\psi_{k_0}^{\infty}(z,\lambda_s)$ *quantized* if the quantization conditions (59) is satisfied just for the number k_0. Of course it is satisfied also for the number $-k_0$ so the solution $\psi_{-k_0}^{\infty}(z,\lambda_s)$ is also quantized and both the solutions coincide up to a constant.

Therefore if we identify the quantized solutions $\psi_{k_0}^{\infty}(z,\lambda_s)$ and $\psi_{-k_0}^{\infty}(z,\lambda_s)$ then an exceptional set for them is just a section $\bigcup_{r\in A} L_{k_0}^r \cap \bigcup_{r\in A} L_{-k_0}^r = \partial D_{k_0} \cap \partial D_{-k_0}$ while for the remaining FS's their exceptional sets are kept unchanged.

We thus come up to the following conclusion:

Corollary 1a. *In the case of the potential $W_n^{\infty}(z)$ when the singular high energy limit $\lambda_s \to \infty$ (or equivalently $s \to \infty$) are considered roots of both the quantized FS's and the not quantized ones are distributed on the $C_{\rm cut}$-plane uniquely on the exceptional SL's corresponding to these solutions and the corresponding formulae of theorem 2b, where λ and Λ should be substituted by λ_s and Λ_s respectively, are valid for these distributions excluding formulae (37) and (38) which are no longer valid for the quantized solutions (see figure 5(c)).*

Exactly the same notes as above can be done with respect to the quantized solutions $\psi_{k_0}(z,\lambda_s) = {\rm i}(-1)^s\psi_{-k_0}(z,\lambda_s)$ corresponding to the potential $W_n(z,\lambda)+1$. They are now both deprived of zeros lying on the infinite SL's emerging from the turning points $z_{k_0}(\lambda)$ and $z_{-k_0}(\lambda)$ while keeping their zeros distributed on the inner SL linking these two turning points and on their remaining exceptional SL's.

It is therefore clear how the corresponding conclusion for the quantized high energy limit for the potential $W_n(z,\lambda)+1$ should sound.

Corollary 1b. *In the singular high energy limit $\lambda_s \to \infty$ roots of FS's for the potential $W_n(z,\lambda)+1$ are distributed uniquely on exceptional lines corresponding to these solutions according to the formulae of theorems 3b, where λ and Λ should be substituted by λ_s and Λ_s respectively, except the quantized solutions $\psi_{k_0}(z,\lambda_s) = {\rm i}(-1)^s\psi_{-k_0}(z,\lambda_s)$ for which the formulae (56) and (57) are no longer valid (see figure 5(d)).*

8. Summary and discussion

In this paper we have shown that the high energy limit distributions of zeros of appropriately scaled fundamental solutions (FS) for polynomial potentials can be described completely both for the quantized and non-quantized cases of these solutions. The quantized cases were considered earlier by Eremenko *et al* [9] where the authors pointed out that the exceptional Stokes lines (ESL) are the loci of zeros of FS's. We have completed their observation in this case giving a detailed description of positions of these zeros on the ESL's. However we have considered the zeros distribution problem of FS's in general showing that loci of zeros of FS's on their ESL's is their common property which has to be completed by theorems 2b and 3b which find additional zeros of FS's outside the ESL's for the critical cases of SG's.

However, to get stable patterns for zeros loci distributions of FS's we have been forced to consider regular asymptotic high energy limit of these loci getting as a result island pictures for these distributions with the islands numbered by q in the corresponding theorems. Different regular limits (controlled by the Λ and R parameters) have lead to different distributions of zeros inside each island with the latter distributions being controlled by the r-parameter in theorems 2–3.

It is worth noting also that taking these regular limits stabilizes the limit zeros distributions of FS's in the same way as the rescaling z-variable by energy E in the initial polynomial potential $P_n(z)$ stabilizes its zeros distribution in the limit $E \to \infty$ reducing it to the zeros loci of the polynomial $(-{\rm i}\alpha z)^n - 1$.

Acknowledgments

I would like to thank to Boris Shapiro for focusing my attention on the problem of zeros of FS's as well as for many discussions on it during my visit in the Department of Mathematics of Stockholm University.

Appendix A

We shall show here that the integrals (8) are all pure imaginary. We simply calculate them. To this goal it is necessary to consider $W_n^\infty(z)$ on the cut z-plane. We make all these cuts from each turning point z_k parallely to the real axis and running to $\mathrm{Re}\, z = -\infty$ if z_k lies to the left from the imaginary axis or is placed on it and running to $\mathrm{Re}\, z = +\infty$ in the opposite case. Then argument of $z - z_k$ will be taken from the interval $(-\pi, \pi)$ for the first group of turning points taking the values $-\pi, \pi$ below and above the corresponding cuts respectively (see figure 1(*a*)). For the second group of z_k their arguments are taken from the interval $(0, 2\pi)$ with the values $0, 2\pi$ above and below the cuts, respectively.

Now, making in the integral (8) the following change of variable $z \to \xi = (-\mathrm{i}\alpha z)^n$ we are to calculate the integrals

$$\int_{z_k}^{z_{-k}} \sqrt{W_n^\infty(z)}\,\mathrm{d}z = \int_{C_k} \sqrt{W_n^\infty(z)}\,\mathrm{d}z = -\frac{1}{\mathrm{i}\alpha n}\int_{\tilde{C}_k} (\xi - 1)^{\frac{1}{2}} \xi^{\frac{1-n}{n}}\,\mathrm{d}\xi, \tag{A.1}$$

where the integration contour C_k is an arc of a unit radius starting from the point z_k and ending at $z = z_{-k}$, running clockwise and avoiding all the met turning points $z_{k-1}, \ldots, z_{-k+1}$ (see figure 1(*a*)) while $\tilde{C}_k$ is the image of C_k on the ξ-Riemann surface defined by the above change of variable (see figure A1).

It is easy to note that $\tilde{C}_k$ is a collection of $2k$ or $2k-1$ unit radius circles on this ξ-Riemann surface depending on α. The first value corresponds to $\alpha = 1$ and the second to $\alpha = \mathrm{e}^{-\frac{\mathrm{i}\pi}{n}}$. The first circle starts at the point $\xi = 1$ on the sheet on which $\arg \xi = 2k\pi$. The latter argument is just that of $\xi_k = (-\mathrm{i}\alpha z_k)^n$. In figure A1 this is the kth sheet from above with a_{k+1} as the lower boundary of the cut between 0 and 1 on this sheet. Each next circle is an image of successive arcs of the contour C_k. The last circle corresponds to the unit radius arc between the points z_{-k+1} and z_{-k} and it ends at unity on the $n - k + 1$-th sheet.

As it follows from figure A1 the integration along $\tilde{C}_k$ on the ξ-Riemann surface can be deformed to go along the cuts between the points $\xi = 0$ and $\xi = 1$ on the respective sheets of the surface. Then the integrations along the a_{k+1} and a_{n-k+1} cuts survive only (the remaining ones mutually cancel) so that their contributions to the last integral in (A.1) (call it I_k) are as follows:

$$\begin{aligned}
I_k &= -\frac{1}{\mathrm{i}\alpha n}\int_{\tilde{C}_k} (\xi - 1)^{\frac{1}{2}} \xi^{\frac{1-n}{n}}\,\mathrm{d}\xi \\
&= \left\{\begin{array}{ll}
-\dfrac{1}{\mathrm{i}n}\displaystyle\int_0^1 (1-\xi)^{\frac{1}{2}} \xi^{\frac{1-n}{n}}\,\mathrm{d}\xi \left(-\mathrm{e}^{\mathrm{i}\frac{n\pi+4k\pi}{2n}} + \mathrm{e}^{\mathrm{i}\frac{n\pi+4\pi(n-k+1)}{2n}}\right) & \text{for} \quad \alpha = 1 \\
-\dfrac{1}{\mathrm{i}\alpha n}\displaystyle\int_0^1 (1-\xi)^{\frac{1}{2}} \xi^{\frac{1-n}{n}}\,\mathrm{d}\xi \left(-\mathrm{e}^{\mathrm{i}\frac{n\pi+4(k-1)\pi}{2n}} + \mathrm{e}^{\mathrm{i}\frac{n\pi+4\pi(n-k)}{2n}}\right) & \text{for} \quad \alpha = \mathrm{e}^{-\frac{\mathrm{i}\pi}{n}}
\end{array}\right\} \\
&= \left\{\begin{array}{ll}
\dfrac{2\mathrm{i}}{n}\sin\dfrac{2k\pi}{n} B\left(\dfrac{3}{2}, \dfrac{1}{n}\right) & \text{for} \quad \alpha = 1 \\
\dfrac{2\mathrm{i}}{n}\sin\dfrac{(2k-1)\pi}{n} B\left(\dfrac{3}{2}, \dfrac{1}{n}\right) & \text{for} \quad \alpha = \mathrm{e}^{-\frac{\mathrm{i}\pi}{n}}
\end{array}\right\} \qquad k \in A_{n,\alpha}
\end{aligned} \tag{A.2}$$

where $B(x, y)$ is the beta function.

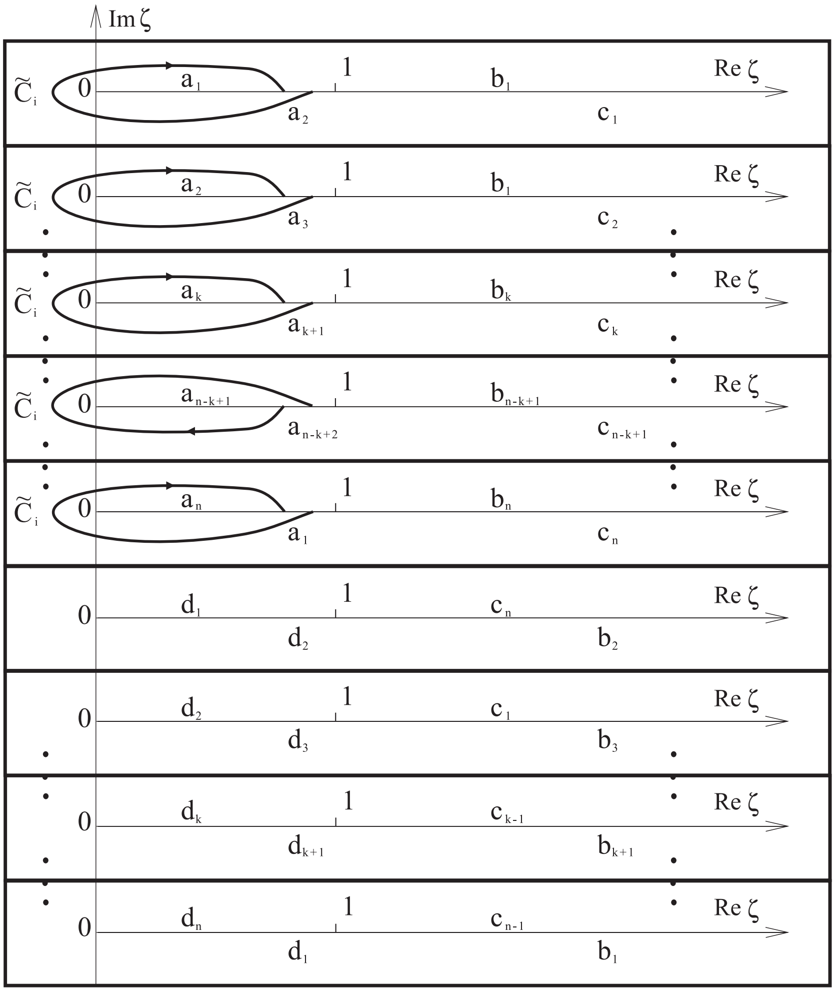

Figure A1. $2n$ cut ξ-planes of Riemann surface of $(\xi - 1)^{\frac{1}{2}}\xi^{\frac{1-n}{n}}$. The letters $a_1, \ldots, d_n$ denote boundaries of cuts. Boundaries with the same letters are glued.

Appendix B

Here we prove the lemma of section 5, theorems 2a–2b of section 5 and lemma' of section 6.

Proof of lemma. To prove this lemma let us note that $\omega_\infty(z)$ is holomorphic in $C_{\mathrm{cut}}(\epsilon)$ vanishing there as $z^{-\frac{1}{2}n-2}$ for $z \to \infty$. Therefore the integrals $I_k(z) = \int_{\gamma_k(z)} |\omega_\infty(\xi)\,\mathrm{d}\xi|$, $k = 1, \ldots, n+2$, are well defined in the closure $\bar{C}_{\mathrm{cut}}(\epsilon)$ and are bounded there. Hence C_ϵ^∞ does exist and is finite in $\bar{C}_{\mathrm{cut}}(\epsilon)$ and therefore can be taken the same for each $D_{k,\epsilon}$. The estimation (17) follows then directly from (15) since $\left|1 - \mathrm{e}^{-2\sigma_k\lambda\tilde{W}_n^\infty(y_i,y_{i+1})}\right| \leqslant 2$ in this formula for each i if all the integrations are performed on canonical paths $\gamma_k(z)$ what is always possible by the definition of $D_{k,\epsilon}$. □

Proof of lemma'. As we have mentioned in section 6 the turning points $z_k(\lambda)$ of $W_n(z,\lambda)$ tend to corresponding roots z_k of $W_n^\infty(z)$ in the limit $\lambda \to \infty$. Therefore taking $\lambda_0 > 0$ sufficiently large we can find all the roots $z_k(\lambda)$ in the corresponding circles $|z - z_k| < \epsilon$ for $|\lambda| > \lambda_0$.

Therefore as in the case of $\omega_\infty(z)$ also $\omega(z,\lambda)$ corresponding to the potential $W_n(z,\lambda)$ is holomorphic in the closure $\bar{C}_{\mathrm{cut}}(\epsilon)$ and bounded there since it vanishes there as $z^{-\frac{1}{2}n-2}$ for $z\to\infty$. Hence, as previously, the integrals $I_k(z,\lambda)=\int_{\gamma_k(z)}|\omega(\xi,\lambda)\,\mathrm{d}\xi|$, $k\in B_{n,\alpha}$, are well defined in the closure $\bar{C}_{\mathrm{cut}}(\epsilon)$ and are bounded there by $C_\epsilon(\lambda)$ defined by

$$C_\epsilon(\lambda)=\liminf_{\gamma_k(z),z\in\bar{C}_{\mathrm{cut}}(\epsilon),k\in B_{n,\alpha}} I_k(z,\lambda). \tag{B.1}$$

□

However $C_\epsilon(\lambda)$ is a continuous function of $\mathrm{Re}\,\lambda$ and $\mathrm{Im}\,\lambda$ for $|\lambda|>\lambda_0$. Therefore since also $\lim_{\lambda\to\infty}C_\epsilon(\lambda)=C_\epsilon^\infty$ in a closure of $\bar{C}_{\mathrm{cut}}(\epsilon)$, $C_\epsilon(\lambda)$ is bounded by some constant $C_\epsilon(\lambda_0)$ with the property that $C_\epsilon(\lambda_0)\to C_\epsilon^\infty$ for $\lambda_0\to\infty$. The latter means that for λ_0 sufficiently large we can choose a constant $C_\epsilon(\lambda_0)$ to be independent of λ_0 and to differ from C_ϵ^∞ by ϵ, i.e. $C_\epsilon(\lambda_0)\leqslant C_\epsilon=C_\epsilon^\infty+\epsilon$.

Now the estimation (41) follows directly from (14) written for the potential $W_n(z,\lambda)+1$.

Proof of theorems 2a and 2b. According to lemma, for every k, $\psi_k^\infty(z,\lambda)$ can be represented in $D_{k,\epsilon}$ for $|\lambda|>\lambda_0\gg C_{k,\epsilon}$ as

$$\psi_k^\infty(z,\lambda)=\left(W_n^\infty(z)\right)^{-\frac{1}{4}}\mathrm{e}^{\sigma_k\lambda\int_{z_k}^z\sqrt{W_n^\infty(y)}\,\mathrm{d}y}(1+O(|\lambda|^{-\gamma})),\ \gamma\geqslant 1. \tag{B.2}$$

□

In fact the above estimation of $\psi_k^\infty(z,\lambda)$ in $D_{k,\epsilon}$ can be extended to any finite order in λ^{-1}. It will be convenient to include into this estimation all orders of λ^{-1}, i.e. to represent $\psi_k^\infty(z,\lambda)$ by its full asymptotic form (18). Therefore we will use further the following asymptotic representations for $\psi_k^\infty(z,\lambda)$:

$$\begin{aligned}\psi_k^\infty(z,\lambda)\sim\psi_k^{\infty(as)}(z)&=\left(W_n^\infty(z)\right)^{-\frac{1}{4}}\ \mathrm{e}^{\sigma_k\lambda\int_{z_k}^z\sqrt{W_n^\infty(y)}\,\mathrm{d}y}\chi_k^{\infty(as)}(z,\lambda)\\&=\left(W_n^\infty(z)\right)^{-\frac{1}{4}}\ \mathrm{e}^{\sigma_k\lambda\int_{z_k}^z\sqrt{W_n^\infty(y)}\,\mathrm{d}y+\int_{\infty_k}^z Z_k^\infty(y,\lambda)\,\mathrm{d}y}.\end{aligned} \tag{B.3}$$

Despite the fact that the asymptotic semiclassical series $Z_k^\infty(y,\lambda)$ is generally divergent we can use it in its full form having however in mind that it can be abbreviated in any moment at some finite power of λ^{-1} to provide us with an estimation like (B.2) but valid then up to the order kept.

To perform detailed calculations we shall assume the C_{cut} plane to be cut in the following way.

For the root z_0 and $z_{\frac{n}{2}}$ (if they are) the corresponding cuts coincide with the positive and negative parts of the imaginary axis, respectively. So there are in fact two SL's, mutually parallel to themselves on C_{cut} and lying on different sites of the cut.

For the remaining roots the corresponding cuts are parallel to the real axis running to the left for the roots lying on the left from the imaginary axis and running to the right for the roots placed on the right from the imaginary axis so that the pattern of cuts is completely symmetric with respect to the imaginary axis.

Arguments of differences $z-z_k$, $k\in A_{n,\alpha}$, are always taken with respect to axis emerging from the roots z_k parallely to the real axis and keeping its direction (see figure 1(*a*)). For the right-hand roots these arguments are taken from above the cuts. The arguments are positive if they are taken clockwise and negative in the opposite cases.

Arranging cuts in the above way makes C_{cut} simple connected what makes all considered functions defined on it single valued as well as it allows for simple and unique calculations of arguments of respective quantities.

With these conventions a sign of $\mathrm{Re}\left(\lambda\int_{z_k}^z\sqrt{W_n^\infty(y)}\,\mathrm{d}y\right)$, $|\arg\lambda|\leqslant\frac{\pi}{n}$, is therefore always positive for the sectors S_0, $S_{\pm\frac{n+3}{2}}$ and $S_{\pm\frac{n+2}{2}}$. This sign is always negative for the sectors $S_{\frac{n+1}{2}}$

and $S_{\pm\frac{n}{2}}$. For the remaining sectors S_k, $k \in B_{n,\alpha}$, this sign is negative above the cuts crossing these sectors and positive below them, i.e. it changes each time these cuts are crossed (see figure 1(*a*)). Note that this switch of the signs does not occur on the other cuts. In the same way behave all terms of the expansion Z_k^∞ in (B.3).

Also the factor $\left(W_n^\infty(z)\right)^{-\frac{1}{4}}$ in (B.3) crossing each cut changes its phase by $\mp i$. The upper sign has to be taken when the cut on the positive imaginary axis is crossed from its right or the left-hand cuts are crossed from above while the lower has to be taken for the negative imaginary axis cut and the r.h. cuts crossed in the same directions (i.e. to the left and down, respectively). The signs are changed to opposite if the cuts are crossed in the opposite directions (see figures 1(*a*) and (*b*)).

Therefore we assume the form (B.3) of the solution $\psi_k^\infty(z,\lambda)$ to be valid in the negative parts of the sectors S_k, $k = \pm 1, \pm 2, \ldots, k \in B_{n,\alpha}$, and in the sectors $S_{\frac{n+1}{2}}$ and $S_{\pm\frac{n+2}{2}}$, so that $\sigma_k = +1$ there.

The form (B.3) is also assumed to be valid for the sectors S_0, $S_{\pm\frac{n+1}{2}}$ and $S_{\pm\frac{n}{2}}$ but then $\sigma_k = -1$ for the corresponding solutions.

The asymptotic forms (B.3) of the FS's when the latter are continued analytically across the C_{cut}-plane along canonical paths while keeping their validity change therefore appropriately by acquiring additional phases by the factor $\left(W_n^\infty(z)\right)^{-\frac{1}{4}}$ or by switching their signs by the exponentials $\lambda\int_{z_k}^z \sqrt{W_n^\infty(y)}\,\mathrm{d}y + \int_{\infty_k}^z Z_k^\infty(y,\lambda)\,\mathrm{d}y$ on the cuts met.

We shall consider below in detail the analytic continuation of the exact solutions $\psi_k^\infty(z,\lambda)$ and their asymptotic expansions (B.3) as well along canonical paths. We will get first exact results of such continuations to come over next to their asymptotic forms and to get finally their limit for $\lambda \to \infty$. We shall consider them one by one beginning with $k = 0, \frac{n+3}{2}, \frac{n+2}{2}, 1, 2, \ldots, \frac{n+1}{2}$, i.e. going down the corresponding SG's, but neglecting detailed calculations for the solutions for which the corresponding results can be obtained be symmetry arguments.

$k = 0$ *case*. This case corresponds to an even-n and $\alpha = \mathrm{e}^{-\frac{i\pi}{n}}$ so that the corresponding SG with exceptional SL's for this case is shown in figures 1(*d*) and (*e*).

We shall continue $\psi_0^\infty(z,\lambda)$ to vicinities $V_{k,\epsilon}$ of its exceptional SL's (ESL) expressing it simply as a linear combinations of FS's defined in sectors closest to the particular ESL considered at the moment. By symmetry arguments we can do it only for the left ESL's of figures 1(*d*) and (*e*).

Using first the exact forms (13) of the solutions we can get coefficients of the corresponding linear combinations also in their limit forms up to any order of λ^{-1} by substituting the exact expressions by their asymptotics (B.3).

Therefore continuing $\psi_0^\infty(z,\lambda)$ to the vicinity of the ESL emerging from the root z_k, $k = 1, \ldots, \frac{n}{2}$ we get

$$\begin{aligned}
\psi_0^\infty(z,\lambda) &= \mathrm{i}\,\mathrm{e}^{-\lambda\int_{z_0}^{z_k}\sqrt{W_n^\infty(y)}\,\mathrm{d}y}\frac{\chi_{0\to k+1}^\infty}{\chi_{k\to k+1}^\infty}\psi_k^\infty(z,\lambda) \\
&\quad - \mathrm{i}\,\mathrm{e}^{-\lambda\int_{z_0}^{z_k}\sqrt{W_n^\infty(y)}\,\mathrm{d}y+\lambda\int_{z_k}^{z_{k+1}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y}\frac{\chi_{0\to k}^\infty}{\chi_{k\to k+1}^\infty}\psi_{k+1}^\infty(z,\lambda) \\
&= \frac{\left(W_n^\infty(z)\right)^{-\frac{1}{4}}\mathrm{e}^{-\lambda\int_{z_0}^{z_k}\sqrt{W_n^\infty(y)}\,\mathrm{d}y}}{\chi_{k\to k+1}^\infty}\Big(\mathrm{e}^{-\lambda\int_{z_k}^{z}\sqrt{W_n^\infty(y)}\,\mathrm{d}y}\chi_{0\to k+1}^\infty\chi_k^\infty(z,\lambda) \\
&\quad - \mathrm{i}\,\mathrm{e}^{\lambda\int_{z_k}^{z}\sqrt{W_n^\infty(y)}\,\mathrm{d}y}\chi_{0\to k}^\infty\chi_{k+1}^\infty(z,\lambda)\Big),
\end{aligned} \tag{B.4}$$

where $\chi_{i\to j}^\infty = \lim_{z\to\infty_j}\chi_i^\infty(z) = \lim_{z\to\infty_i}\chi_j^\infty(z) = \chi_{j\to i}^\infty$ (see, for example [4], ref. 2).

From (B.4) we get the following exact condition for loci $\zeta_{k,m}^{(0)}$ of zeros of $\psi_0^\infty(z,\lambda)$ on the considered ESL

$$\int_{z_k}^{\zeta_{k,m}^{(0)}} \sqrt{W_n^\infty(y)}\,\mathrm{d}y = +\left(m-\frac{1}{4}\right)\frac{\mathrm{i}\pi}{\lambda} + \frac{1}{2\lambda}\ln\frac{\chi_{0\to k+1}^\infty \chi_k^\infty\left(\zeta_{k,m}^{(0)},\lambda\right)}{\chi_{0\to k}^\infty \chi_{k+1}^\infty\left(\zeta_{k,m}^{(0)},\lambda\right)}, \tag{B.5}$$

where m is a positive integer and the choice of the '+'-sign is determined by the positiveness of the lhs of (B.5) on the ESL considered.

Now we would like to come over in (B.5) to the limit $\lambda \to \infty$. However if we want to use the results of lemma we need $\zeta_{k,m}^{(0)}$ to be kept inside the domain $D_{k,\epsilon}$, i.e. their distances to z_k in the limit $\lambda \to \infty$ should be greater than ϵ. It then follows from (B.5) that for large λ also m should be large, i.e.

$$\frac{m\pi}{|\lambda|} > \limsup_{|\phi|\leqslant\pi}\left|\int_{z_k}^{z_k+\epsilon \mathrm{e}^{\mathrm{i}\phi}} \sqrt{W_n^\infty(y)}\,\mathrm{d}y\right| \equiv I_k(\epsilon). \tag{B.6}$$

This condition means that we have to take into account only those zeros $\zeta_{k,m}^{(0)}$ which are sufficiently far from z_k. Therefore for those which can fall onto z_k we have to take m sufficiently large, i.e. $m > m_0 = |\lambda| I_k(\epsilon)/\pi$.

For the remaining ones, i.e. those which are to approach finite nonzero distances to z_k it is necessary to make m growing linearly with λ, i.e. $m = q[|\lambda|] + r$ with $q = 1, 2, \ldots, r = 0, \pm 1, \pm 2, \ldots$ and $[|\lambda|]$ being the step function of $|\lambda|$, i.e. $|\lambda| = [|\lambda|] + \Lambda, 0 \leqslant \Lambda < 1$.

Let us note further that the latter case will cover also the previous one when $q = 0$ and $r > m_0$. In all the formulae below we shall assume the cases of zeros falling down onto z_k to be taken into account just in this way.

Having this in mind we can come in (B.5) to the limit $[|\lambda|] \to \infty$ to get for $\zeta_{k,m}^{(0)} \sim \zeta_{k,qr}^{(0)}(\lambda)$

$$\int_{z_k}^{\zeta_{k,qr}^{(0)}(\lambda)} \sqrt{W_n^\infty(y)}\,\mathrm{d}y = +\left(q[|\lambda|] + r - \frac{1}{4}\right)\frac{\mathrm{i}\pi}{\lambda} + \frac{1}{2\lambda}\int_{K_k(\zeta_{k,qr}^{(0)}(\Lambda))} Z_0^\infty(y,\lambda)\,\mathrm{d}y, \tag{B.7}$$

where $K_k(\zeta_{k,qr}^{(0)}(\lambda))$ is a contour shown in figure 4(a) which starts and ends at the point $\zeta_{k,qr}^{(0)}(\Lambda)$. This contour is not closed since it starts and ends at different sites of the cut emerging from z_k.

In fact (B.7) is an implicit condition for the asymptotic expansion $\zeta_{k,qr}^{(0)}(\lambda)$ of zeros $\zeta_{k,m}^{(0)}$ as $\lambda \to \infty$. Therefore we can look for the following form of semiclassical expansion for $\zeta_{k,qr}^{(0)}(\lambda)$

$$\zeta_{k,qr}^{(0)}(\lambda) = \sum_{p\geqslant 0} \frac{1}{\lambda^p}\zeta_{k,qrp}^{(0)}(\Lambda) \tag{B.8}$$

with $\zeta_{k,0r0}^{(0)}(\Lambda) \equiv z_k$.

Let us now note that the condition (B.7) can be written uniformly as

$$\int_{K_k(\zeta_{k,qr}^{(0)}(\lambda))}\left(\frac{1}{2}\sqrt{W_n^\infty(y)} - \frac{1}{2\lambda}Z_0^\infty(y,\lambda)\right)\mathrm{d}y = +\left(q[|\lambda|] + r - \frac{1}{4}\right)\frac{\mathrm{i}\pi}{\lambda}. \tag{B.9}$$

We can now calculate the asymptotic series (B.8) in the limit $[|\lambda|] \to \infty$ with fixed Λ from the following formula:

$$\begin{aligned}
&\int_{K_k(\zeta_{k,qr0}^{(0)}(\Lambda))}\left(\frac{1}{2}\sqrt{W_n^\infty(y)} - \frac{1}{2\lambda}Z_0^\infty(y,\lambda)\right)\mathrm{d}y \\
&\qquad + 2\sum_{s\geqslant 1}\frac{1}{s!}\left.\left(\frac{1}{2}\sqrt{W_n^\infty(y)} - \frac{1}{2\lambda}Z_0^\infty(y,\lambda)\right)^{(s)}\right|_{y=\zeta_{k,qr0}^{(0)}(\Lambda)}\left(\sum_{p\geqslant 1}\frac{1}{\lambda^p}\zeta_{k,qrp}^{(0)}(\Lambda)\right)^s \\
&\quad = \left(q[|\lambda|] + r - \frac{1}{4}\right)\frac{\mathrm{i}\pi}{\lambda}, \qquad q > 0
\end{aligned} \tag{B.10}$$

and with a similar formula for $q = 0$ obtained from the last one where $\zeta^{(0)}_{k,qr0}(\Lambda)$ is substituted by $\zeta^{(0)}_{k,0r1}(\Lambda)/\lambda$ and $p > 1$.

The limit $[|\lambda|] \to \infty$ with fixed Λ (and $\arg\lambda \equiv \beta, |\beta| < \pi/n$) considered above is of course regular.

Therefore for $q > 0$ the zero term $\zeta^{(0)}_{k,qr0}(\Lambda)$ of such an expansion is given by the equation

$$\int_{K_k(\zeta^{(0)}_{k,qr0}(\Lambda))} \frac{1}{2}\sqrt{W_n^\infty(y)}\,\mathrm{d}y = \int_{z_k}^{\zeta^{(0)}_{k,qr0}(\Lambda)} \sqrt{W_n^\infty(y)}\mathrm{d}y = q\mathrm{i}\pi\,\mathrm{e}^{-\mathrm{i}\beta} \tag{B.11}$$

which shows that this term is independent of Λ.

The next term is given explicitly by

$$\zeta^{(0)}_{k,qr1}(\Lambda) = \left(r - q\Lambda - \frac{1}{4}\right)\frac{\mathrm{i}\pi\,\mathrm{e}^{-\mathrm{i}\beta}}{\sqrt{W_n^\infty(\zeta^{(0)}_{k,qr0}(\Lambda))}} \tag{B.12}$$

and so on.

For $q = 0$ we get correspondingly

$$\int_{z_k}^{z_k+\zeta^{(0)}_{k,0r1}(\Lambda)/\lambda} \sqrt{W_n^\infty(y)}\,\mathrm{d}y = \left(r - \frac{1}{4}\right)\frac{\mathrm{i}\pi}{\lambda}, \qquad r > m_0 = |\lambda| I_k(\epsilon)/\pi \tag{B.13}$$

and

$$\zeta^{(0)}_{k,0r2}(\Lambda) = \frac{1}{8}\frac{\int_{K_k(z_k+\zeta^{(0)}_{k,0r1}(\Lambda)/\lambda)} X_1^\infty(y)\,\mathrm{d}y}{\sqrt{W_n^\infty\big(z_k + \zeta^{(0)}_{k,0r1}(\Lambda)/\lambda\big)}} \tag{B.14}$$

$k = \frac{n+3}{2}$ *case.*

This case corresponds to n odd and to the pattern of ESL's shown in figure 1(*a*). We have to continue $\psi^\infty_{\frac{n+3}{2}}(z,\lambda)$ to vicinities of ESL's emerging from the root z_0 and from the roots $z_k, k = \pm 1, \pm 2, \ldots, \pm\frac{n-1}{2}$. In the first case we have to express $\psi^\infty_{\frac{n+3}{2}}(z,\lambda)$ linearly by the solutions $\psi^\infty_{-\frac{n+3}{2}}(z,\lambda)$ and $\psi^\infty_{-1}(z,\lambda)$. For the remaining ESL's $\psi^\infty_{\frac{n+3}{2}}(z,\lambda)$ is continued in the same way as in the previous case. We have therefore

$$\psi^\infty_{\frac{n+3}{2}}(z,\lambda) = \frac{\chi^\infty_{\frac{n+3}{2}\to -1}}{\chi^\infty_{-1\to-\frac{n+3}{2}}}\psi^\infty_{-\frac{n+3}{2}}(z,\lambda) + \mathrm{i}\,\mathrm{e}^{\lambda\int_{z_0}^{z_{-1}}\sqrt{W_n^\infty(y)}\mathrm{d}y}\frac{\chi^\infty_{\frac{n+3}{2}\to-\frac{n+3}{2}}}{\chi^\infty_{-1\to-\frac{n+3}{2}}}\psi^\infty_{-1}(z,\lambda) \tag{B.15}$$

for the first ESL and

$$\begin{aligned}\psi^\infty_{\frac{n+3}{2}}(z,\lambda) = {} & \mathrm{i}\,\mathrm{e}^{-\lambda\int_{z_0}^{z_k}\sqrt{W_n^\infty(y)}\mathrm{d}y}\frac{\chi^\infty_{\frac{n+3}{2}\to k+1}}{\chi^\infty_{k\to k+1}}\psi^\infty_k(z,\lambda)\\ & - \mathrm{i}\,\mathrm{e}^{-\lambda\int_{z_0}^{z_k}\sqrt{W_n^\infty(y)}\mathrm{d}y+\lambda\int_{z_k}^{z_{k+1}}\sqrt{W_n^\infty(y)}\mathrm{d}y}\frac{\chi^\infty_{\frac{n+3}{2}\to k}}{\chi^\infty_{k\to k+1}}\psi^\infty_{k+1}(z,\lambda)\end{aligned} \tag{B.16}$$

for the ESL's emerging from the roots $z_k, k = 1, 2, \ldots, \frac{n-1}{2}$ while for the ES's emerging from the roots with the opposite sign of k we get

$$\begin{aligned}\psi^\infty_{\frac{n+3}{2}}(z,\lambda) = {} & \mathrm{e}^{-\lambda\int_{z_0}^{z_{-k}}\sqrt{W_n^\infty(y)}\mathrm{d}y}\frac{\chi^\infty_{\frac{n+3}{2}\to-k-1}}{\chi^\infty_{-k\to-k-1}}\psi^\infty_{-k}(z,\lambda)\\ & + \mathrm{i}\,\mathrm{e}^{-\lambda\int_{z_0}^{z_{-k}}\sqrt{W_n^\infty(y)}\mathrm{d}y+\lambda\int_{z_{-k}}^{z_{-k-1}}\sqrt{W_n^\infty(y)}\mathrm{d}y}\frac{\chi^\infty_{\frac{n+3}{2}\to-k}}{\chi^\infty_{-k\to-k-1}}\psi^\infty_{-k-1}(z,\lambda).\end{aligned} \tag{B.17}$$

Arguing in exactly the same way as in the previous case from (B.15) and (B.17) for zeros of $\psi^\infty_{\frac{n+3}{2}}(z,\lambda)$ on the ESL's emerging from the roots $z_{\pm k}, k = 0, 1, 2, \ldots, \frac{n-1}{2}$ we get the corresponding conditions

$$\int_{K_{\pm k}(\zeta^{(\frac{n+3}{2})}_{\pm k,qr}(\Lambda))}\left(\frac{1}{2}\sqrt{W_n^\infty(y)}-\frac{1}{2\lambda}Z^\infty_{\frac{n+3}{2}}(y,\lambda)\right)\mathrm{d}y=\pm\left(q[|\lambda|]+r-\frac{1}{4}\right)\frac{\mathrm{i}\pi}{\lambda} \qquad \text{(B.18)}$$

$k=\frac{n+2}{2}$ *case.*

This case corresponds to the even n and $\alpha = 1$, see figure 1(*b*). We have to continue $\psi^\infty_{\frac{n+2}{2}}(z,\lambda)$ to vicinities of its ESL's emerging from the roots $z_{\pm k}$, $k = 0, 1, \ldots, \frac{n-2}{2}$ and from $z_{\frac{n}{2}}$.

However, the corresponding formulae for the positions of zeros on the ESL's emerging from all the roots mentioned above except the last one coincide for the obvious reasons exactly with the respective formulae (B.18) while for the ESL's emerging from the last root we have

$$\psi^\infty_{\frac{n+2}{2}}(z,\lambda)=\mathrm{i}\,\mathrm{e}^{-\lambda\int_{z_0}^{z_{\frac{n}{2}}}\sqrt{W_n^\infty(y)}\mathrm{d}y}\left(\chi^\infty_{\frac{n+3}{2}\to-\frac{n}{2}}\psi^\infty_{\frac{n}{2}}(z,\lambda)-\chi^\infty_{\frac{n+3}{2}\to\frac{n}{2}}\psi^\infty_{-\frac{n}{2}}(z,\lambda)\right), \qquad \text{(B.19)}$$

where z is assumed to lie on the lhs of the cut emerging from $z_{\frac{n}{2}}$.

Therefore for zeros $\zeta^{(\frac{n+2}{2})}_{\frac{n}{2},qr}$ of $\psi^\infty_{\frac{n+2}{2}}(z,\lambda)$ lying in the vicinity of the ESL emerging from $z_{\frac{n}{2}}$ we get

$$\int_{K_{\frac{n}{2}}(\zeta^{(\frac{n+2}{2})}_{\frac{n}{2},qr}(\Lambda))}\left(\frac{1}{2}\sqrt{W_n^\infty(y)}-\frac{1}{2\lambda}Z^\infty_{\frac{n+2}{2}}(y,\lambda)\right)\mathrm{d}y=\pm\left(q[|\lambda|]+r-\frac{1}{4}\right)\frac{\mathrm{i}\pi}{\lambda}, \qquad \text{(B.20)}$$

where $\pm$ corresponds to the positions of $\zeta^{(\frac{n+2}{2})}_{\frac{n}{2},qr}$ on the left or right from the cut, respectively. Of course we always get the same values for $\zeta^{(\frac{n+2}{2})}_{\frac{n}{2},qr}$ irrespective of the site chosen.

$k=1,2,\ldots,\frac{n}{2}$ *cases*

We shall consider all the enumerated cases jointly since the corresponding roots are all on the left from the imaginary axis independent of the parity of n.

We have to distinguish two cases corresponding to $\arg\lambda = \beta$ is equal to 0 or differs from it. In the second case (the corresponding SG is shown in figure 2) the procedure of analytical continuations of $\psi^\infty_k(z,\lambda)$ along canonical paths to vicinities of all its the ESL's is completely analogous to the previous cases since all the sectors having these ESL's as their boundaries are available from the sector S_k along canonical paths. The rules of making these continuations are now clear and their results are gathered by the formulae of the type (B.9), (B.18) or (B.20) so we can focus our attention on the real case of λ.

In the case $\beta = 0$ each inner SL emerging from the root z_k, $k = 1, 2, \ldots, \frac{n}{2}$ is exceptional for the corresponding solution $\psi^\infty_k(z,\lambda)$, see figures 1(*a*)–(*e*), together with the two ESL's which close the sector S_{-k} which is not available directly on any canonical paths from the sector S_k.

On the other hand positions of other ESL's corresponding to $\psi^\infty_k(z,\lambda)$ in the considered case is analogous to their positions for $\beta \neq 0$ and their vicinities can be approached by continuations of $\psi^\infty_k(z,\lambda)$ along canonical paths, i.e. positions of zeros of the considered solution on these ELS's can be established exactly in the same way as for the cases investigated so far. Therefore for positions of these zeros on their ESL's we can invoke again the respective formulae (B.9), (B.18) and (B.20).

Consider now positions of possible zeros of $\psi^\infty_k(z,\lambda)$ on the inner line emerging from the root z_k (it ends at z_{-k}) and on the remaining two lines emerging from z_{-k} and running to infinity. For the inner line we can use any pair of linear independent FS's for both of which this line is not exceptional. These can be for example the solutions $\psi^\infty_{-k+1}(z,\lambda)$ and $\psi^\infty_{-k-1}(z,\lambda)$.

They communicate canonically with the solution $\psi_k^\infty(z,\lambda)$ so we get

$$\psi_k^\infty(z,\lambda) = -\mathrm{e}^{-\lambda\int_{z_k}^{z_{-k+1}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y}\frac{\chi^\infty_{k\to-k-1}}{\chi^\infty_{-k+1\to-k-1}}\psi^\infty_{-k+1}(z,\lambda)$$
$$+\mathrm{e}^{\lambda\int_{z_k}^{z_{-k-1}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y}\frac{\chi^\infty_{k\to-k+1}}{\chi^\infty_{-k+1\to-k-1}}\psi^\infty_{-k-1}(z,\lambda). \tag{B.21}$$

From (B.21) we get for the positions of zeros of $\psi_k^\infty(z,\lambda)$ on its inner ESL (z_k, z_{-k})

$$\int_{K_k(\zeta^{(k)}_{k,qr}(\Lambda))}\left(\frac{1}{2}\sqrt{W_n^\infty(y)} - \frac{1}{2\lambda}Z_k^\infty(y,\lambda)\right)\mathrm{d}y = -\left(m-\frac{1}{4}\right)\frac{\mathrm{i}\pi}{\lambda}. \tag{B.22}$$

It follows from (B.11) that q is now bounded by the integral $I_k \equiv \int_{z_k}^{z_{-k}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y$, i.e. $q \leqslant -I_k/(\mathrm{i}\pi)$.

In order to look for possible zeros of $\psi_k^\infty(z,\lambda)$ on its ESL's emerging from z_{-k} we have to continue it further to vicinities of these lines using the solution $\psi_{-k}^\infty(z,\lambda)$ which also communicates canonically with both the solutions $\psi_{-k+1}^\infty(z,\lambda)$ and $\psi_{-k-1}^\infty(z,\lambda)$. Choosing one of them we can express it as linear combinations of the others to get

$$\psi_{-k+1}^\infty(z,\lambda) = \mathrm{e}^{\lambda\int_{z_{-k}}^{z_{-k+1}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y}\chi^\infty_{-k+1\to-k-1}\psi^\infty_{-k}(z,\lambda)$$
$$-\mathrm{e}^{\lambda\int_{z_{-k}}^{z_{-k+1}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y+\lambda\int_{z_{-k}}^{z_{-k-1}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y}\psi^\infty_{-k-1}(z,\lambda) \tag{B.23}$$

and

$$\psi_{-k-1}^\infty(z,\lambda) = \mathrm{e}^{-\lambda\int_{z_{-k}}^{z_{-k-1}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y}\chi^\infty_{-k-1\to-k+1}\psi^\infty_{-k}(z,\lambda)$$
$$-\mathrm{e}^{-\lambda\int_{z_{-k}}^{z_{-k+1}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y-\lambda\int_{z_{-k}}^{z_{-k-1}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y}\psi^\infty_{-k+1}(z,\lambda) \tag{B.24}$$

respectively.

Now substituting subsequently both the above formulae to (B.21) we get the formulae realizing continuations of $\psi_k^\infty(z,\lambda)$ close to the respective ELS's, i.e. we get in this way

$$\psi_k^\infty(z,\lambda) = -\mathrm{e}^{-\lambda\int_{z_k}^{z_{-k}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y}\frac{\chi^\infty_{k\to-k-1}}{\chi^\infty_{-k+1\to-k-1}}\Bigg(\chi^\infty_{-k+1\to-k-1}\psi^\infty_{-k}(z,\lambda)$$
$$-\ \mathrm{e}^{\lambda\int_{z_{-k}}^{z_{-k-1}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y}\left(1+\mathrm{e}^{2\lambda\int_{z_k}^{z_{-k}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y}\frac{\chi^\infty_{k\to-k+1}}{\chi^\infty_{k\to-k-1}}\right)\psi^\infty_{-k-1}(z,\lambda)\Bigg) \tag{B.25}$$

and

$$\psi_k^\infty(z,\lambda) = +\mathrm{e}^{\lambda\int_{z_k}^{z_{-k}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y}\frac{\chi^\infty_{k\to-k+1}}{\chi^\infty_{-k+1\to-k-1}}\Bigg(\chi^\infty_{-k+1\to-k-1}\psi^\infty_{-k}(z,\lambda)$$
$$-\ \mathrm{e}^{-\lambda\int_{z_{-k}}^{z_{-k+1}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y}\left(1+\mathrm{e}^{-2\lambda\int_{z_k}^{z_{-k}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y}\frac{\chi^\infty_{k\to-k-1}}{\chi^\infty_{k\to-k+1}}\right)\psi^\infty_{-k+1}(z,\lambda)\Bigg). \tag{B.26}$$

The first of these formulae is suitable for analysing zeros of $\psi_k^\infty(z,\lambda)$ in a vicinity of the SL being the lower boundary of the sector S_{-k} while the second in vicinity of the SL being its upper boundary.

The formula (B.25) provides us with the following condition for zeros $\zeta^{(k)-}_{-k,m}$ of $\psi_k^\infty(z,\lambda)$ in the limit $\lambda\to\infty$

$$\int_{z_{-k}}^{\zeta^{(k)-}_{-k,m}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y = -\left(m-\frac{1}{4}\right)\frac{\mathrm{i}\pi}{\lambda}+\frac{1}{4\lambda}\left(\lambda\oint_{K_k}\sqrt{W_n^\infty(y)}\,\mathrm{d}y+\oint_{K_k}Z_k^\infty\,\mathrm{d}y\right)$$
$$-\frac{1}{2\lambda}\ln 2\cos\frac{1}{2}\mathrm{Im}\left(\lambda\oint_{K_k}\sqrt{W_n^\infty(y)}\,\mathrm{d}y+\oint_{K_k}Z_k^\infty\,\mathrm{d}y\right)-\frac{1}{2\lambda}\int_{K_{-k}(\zeta^{(k)-}_{-k,m})}Z_k^\infty\,\mathrm{d}y, \tag{B.27}$$

where K_k and $K_{-k}\left(\zeta^{(k)-}_{-k,m}\right)$ are respective contours of integrations.

Putting now $\lambda_{r_k}\oint_{K_k}\sqrt{W_n^\infty(y)}\mathrm{d}y = 2(r_k+R)\mathrm{i}\pi$, $|R|<\frac{1}{2}$ with r_k a non-negative integer, $\lambda_{r_k}=[\lambda_{r_k}]+\Lambda_{r_k}(R)$, $0\leqslant\Lambda_{r_k}<1$ and $m=q[\lambda_{r_k}]+r$ we get from (B.27)

$$\int_{z_{-k}}^{\zeta_{-k,qr}^{(k)-}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y = -\left(q[\lambda_{r_k}]+r-\frac{1}{4}+\frac{R}{2}\right)\frac{\mathrm{i}\pi}{\lambda_{r_k}}+\frac{1}{4\lambda_{r_k}}\oint_{K_k}Z_k^\infty\,\mathrm{d}y \\ -\frac{1}{2\lambda_{r_k}}\ln 2\cos\left(R\pi+\frac{1}{2}\mathrm{Im}\oint_{K_k}Z_k^\infty\,\mathrm{d}y\right)-\frac{1}{2\lambda_{r_k}}\int_{K_{-k}(\zeta_{-k,qr}^{(k)-})}Z_k^\infty\,\mathrm{d}y, \tag{B.28}$$

where the limit taken is regular with fixed R.

Therefore for the first two terms of the semiclassical expansion of $\zeta_{-k,qr}^{(k)-}$ we get

$$\int_{z_{-k}}^{\zeta_{-k,qr0}^{(k)-}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y = -q\mathrm{i}\pi \\ \zeta_{-k,qr1}^{(k)-}(R) = -\left(r-q\Lambda_{r_k}(R)-\frac{1}{4}+\frac{R}{2}+\frac{1}{2}\ln 2\cos(R\pi)\right)\frac{\mathrm{i}\pi}{\sqrt{W_n^\infty\left(\zeta_{-k,qr0}^{(k)-}\right)}}. \tag{B.29}$$

In exactly the same way we get the condition for zeros $\zeta_{-k,m}^{(k)+}$ distributed along the upper SL emerging from the turning point z_{-k} in the limit $\lambda\to\infty$

$$\int_{z_{-k}}^{\zeta_{-k,m}^{(k)+}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y = -\left(m-\frac{1}{4}\right)\frac{\mathrm{i}\pi}{\lambda}+\frac{1}{4\lambda}\left(\lambda\oint_{K_k}\sqrt{W_n^\infty(y)}\,\mathrm{d}y+\oint_{K_k}Z_k^\infty\,\mathrm{d}y\right) \\ +\frac{1}{2\lambda}\ln 2\cos\frac{1}{2}\mathrm{Im}\left(\lambda\oint_{K_k}\sqrt{W_n^\infty(y)}\,\mathrm{d}y+\oint_{K_k}Z_k^\infty\,\mathrm{d}y\right)+\frac{1}{2\lambda}\int_{K_{-k}(\zeta_{-k,m}^{(k)+})}Z_k^\infty\,\mathrm{d}y \tag{B.30}$$

and with the same meaning of notations used we get

$$\int_{z_{-k}}^{\zeta_{-k,qr}^{(k)+}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y = -\left(q[\lambda_{r_k}]+r-\frac{1}{4}+\frac{R}{2}\right)\frac{\mathrm{i}\pi}{\lambda_{r_k}}+\frac{1}{4\lambda_{r_k}}\oint_{K_k}Z_k^\infty\,\mathrm{d}y \\ +\frac{1}{2\lambda_{r_k}}\ln 2\cos\left(R\pi+\frac{1}{2}\mathrm{Im}\oint_{K_k}Z_k^\infty\,\mathrm{d}y\right)+\frac{1}{2\lambda_{r_k}}\int_{K_{-k}(\zeta_{-k,qr}^{(k)+})}Z_k^\infty\,\mathrm{d}y \tag{B.31}$$

with the following first coefficients of the corresponding semiclassical expansion of $\zeta_{-k,qr}^{(k)+}$

$$\int_{z_{-k}}^{\zeta_{-k,qr0}^{(k)+}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y = -q\mathrm{i}\pi \\ \zeta_{-k,qr1}^{(k)+}(R) = -\left(r-q\Lambda_{r_k}(R)-\frac{1}{4}+\frac{R}{2}-\frac{1}{2}\ln 2\cos(R\pi)\right)\frac{\mathrm{i}\pi}{\sqrt{W_n^\infty\left(\zeta_{-k,qr0}^{(k)+}\right)}} \tag{B.32}$$

$k=\frac{n+1}{2}$ *case*. This is the last distinguished case which has to be treated. It corresponds to an odd n of figure 1(*a*) and to the solution $\psi_{\frac{n+1}{2}}^\infty(z,\lambda)$. We should continue the latter solution to its ESL's emerging from the roots z_k, $k=1,\ldots,\frac{n-1}{2}$ and from z_0. Continuing to any ESL of the first group we get

$$\psi_{\frac{n+1}{2}}^\infty(z,\lambda) = \mathrm{e}^{\lambda\int_{z_{\frac{n+1}{2}}}^{z_k}\sqrt{W_n^\infty(y)}\mathrm{d}y}\chi_{\frac{n+1}{2}\to k-1}^\infty\psi_k^\infty(z,\lambda) \\ -\mathrm{e}^{\lambda\int_{z_{\frac{n+1}{2}}}^{z_k}\sqrt{W_n^\infty(y)}\mathrm{d}y-\lambda\int_{z_k}^{z_{k-1}}\sqrt{W_n^\infty(y)}\,\mathrm{d}y}\chi_{\frac{n+1}{2}\to k}^\infty\psi_{k-1}^\infty(z,\lambda)$$

$$= \left(W_n^\infty(z)\right)^{-\frac{1}{4}} e^{\lambda \int_{z_{\frac{n+1}{2}}}^{z_k} \sqrt{W_n^\infty(y)}\, dy} \times \left(e^{\lambda \int_{z_k}^{z} \sqrt{W_n^\infty(y)} dy} \chi^\infty_{\frac{n+1}{2}\to k-1} \chi_k^\infty(z) + i\, e^{-\lambda \int_{z_k}^{z} \sqrt{W_n^\infty(y)} dy} \chi^\infty_{\frac{n+1}{2}\to k} \chi^\infty_{k-1}(z,\lambda)\right) \tag{B.33}$$

and hence for the respective distributions of zeros $\zeta_{k,qr}^{(\frac{n+1}{2})}(\Lambda)$

$$\int_{K_k(\zeta_{k,qr}^{(\frac{n+1}{2})}(\Lambda))} \left(\frac{1}{2}\sqrt{W_n^\infty(y)} - \frac{1}{2\lambda} Z^\infty_{\frac{n+1}{2}}(y,\lambda)\right) dy = \left(q[|\lambda|] + r - \frac{1}{4}\right)\frac{i\pi}{\lambda}. \tag{B.34}$$

For the ESL emerging from z_0 we have

$$\psi^\infty_{\frac{n+1}{2}}(z,\lambda) = i\, e^{\lambda \int_{z_{\frac{n+1}{2}}}^{z_0} \sqrt{W_n^\infty(y)} dy} \left(\chi^\infty_{\frac{n+1}{2}\to\frac{n+3}{2}} \psi^\infty_{\frac{n+3}{2}}(z,\lambda) - \chi^\infty_{\frac{n+1}{2}\to-\frac{n+3}{2}} \psi^\infty_{-\frac{n+3}{2}}(z,\lambda)\right)$$
$$= i\left(W_n^\infty(z)\right)^{-\frac{1}{4}} e^{\lambda \int_{z_{\frac{n+1}{2}}}^{z_0} \sqrt{W_n^\infty(y)} dy} \left(e^{-\lambda \int_{z_0}^{z} \sqrt{W_n^\infty(y)} dy} \chi^\infty_{\frac{n+1}{2}\to\frac{n+3}{2}} \chi^\infty_{\frac{n+3}{2}}(z) + i\, e^{\lambda \int_{z_0}^{z} \sqrt{W_n^\infty(y)} dy} \chi^\infty_{\frac{n+1}{2}\to-\frac{n+3}{2}} \chi^\infty_{-\frac{n+3}{2}}(z)\right) \tag{B.35}$$

and hence

$$\int_{K_0(\zeta_{0,qr}^{(\frac{n+1}{2})}(\Lambda))} \left(\frac{1}{2}\sqrt{W_n^\infty(y)} - \frac{1}{2\lambda} Z^\infty_{\frac{n+1}{2}}(y,\lambda)\right) dy = \left(q[|\lambda|] + r - \frac{1}{4}\right)\frac{i\pi}{\lambda} \tag{B.36}$$

for the respective distributions of zeros on the considered ESL.

Appendix C

We show here that for the potential $W_n(z,\lambda)$ in the limit $\lambda \to \infty$ if there is a unique inner SL of the SG for this potential between the turning points $z_{k_0}(\lambda)$ and $z_{-k_0}(\lambda)$ then $\arg\lambda \neq 0$ and in the general case $\arg\lambda \sim |\lambda|^{-\frac{2}{n+2}}$. To see this consider in the limit $\lambda \to \infty$ the integral $\lambda \int_{z_{k_0}(\lambda)}^{z_{-k_0}(\lambda)} \sqrt{W_n(y,\lambda) dy}$. With the accuracy $O\left(|\lambda|^{-\frac{3}{n+2}}\right)$ we have

$$\lambda \int_{z_{k_0}(\lambda)}^{z_{-k_0}(\lambda)} \sqrt{W_n(y,\lambda)}\, dy = \lambda \int_{z_{k_0}}^{z_{-k_0}} \sqrt{W_n^\infty(y)}\, dy - \frac{1}{2n}\lambda b'_{n-1}(-i\alpha)^{n-2} \int_{z_{k_0}}^{z_{-k_0}} \frac{y^{n-1}\, dy}{\sqrt{W_n^\infty(y)}} \frac{1}{\lambda^{\frac{2}{n+2}}}, \tag{C.1}$$

where we have used a relation $z_k(\lambda) = z_k - i\frac{1}{n\alpha} b'_{n-1}\lambda^{-\frac{2}{n+2}} + O(\lambda^{-\frac{4}{n+2}})$ valid for any turning point in this limit, with z_k being a turning point of $W_n^\infty(z)$.

According to our assumption we have further ($\lambda = |\lambda|\, e^{i\beta}$)

$$\mathrm{Re}\left(i\beta \int_{z_{k_0}(\lambda)}^{z_{-k_0}(\lambda)} \sqrt{W_n(y,\lambda)} dy\right) = -\frac{1}{2n}\mathrm{Re}\left(b'_{n-1}(-i\alpha)^{n-2} \int_{z_{k_0}}^{z_{-k_0}} \frac{y^{n-1}}{\sqrt{W_n^\infty(y) dy}}\right) \frac{1}{|\lambda|^{\frac{2}{n+2}}} \tag{C.2}$$

and hence finally

$$\beta = -\frac{1}{2n i I_{k_0}}\mathrm{Re}\left(b'_{n-1}(-i\alpha)^{n-2} \int_{z_{k_0}}^{z_{-k_0}} \frac{y^{n-1}}{\sqrt{W_n^\infty(y)}\, dy}\right) \frac{1}{|\lambda|^{\frac{2}{n+2}}} \tag{C.3}$$

where $I_{k_0} = \int_{z_{k_0}}^{z_{-k_0}} \sqrt{W_n^\infty(y)}\, dy$.

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